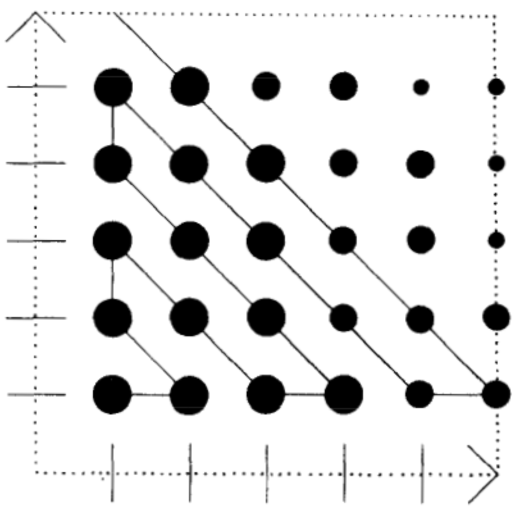
Volume 4

Studies in mathematics education

The education of secondary school teachers of mathematics

Edited by Robert Morris

The teaching of basic sciences Mathematics



Unesco

The teaching of basic sciences

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The education of secondary school teachers of mathematics

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Preface

This fourth Volume of *Studies in Mathematics Education* is devoted to the education, training and support of teachers of mathematics in secondary schools. It continues the theme of teacher education, which began in Volume 3, by concentrating upon the needs of primary school teachers of mathematics. Both volumes are the outcome of a meeting held at Unesco in 1980 which examined the question: does the teaching of mathematics correspond to the needs of the majority of pupils and of their society? From this, teacher education emerged as a priority area in improving mathematics teaching, and this led to the two Volumes being produced.

Like its predecessor, this Volume is intended for teacher educators, mathematics educators and Ministry of Education officials. In addition, it contains a number of sections which will be of interest to teachers of mathematics in secondary schools.

This Volume, like the three earlier ones, was edited by Robert Morris. Unesco wishes to express its appreciation of his work as well as that of the many contributors. The views expressed are, those of the authors; they are not necessarily those of Unesco nor of the editor.

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Introduction

This, the fourth Volume of Studies in Mathematics Education, sustains the theme of the third, namely, teacher education. But while the third Volume was exclusively concerned with the initial and the in-service training of teachers of mathematics in primary schools, this volume concentrates on the parallel needs of secondary school teachers of mathematics. Taken together, Volume 3 and 4 have sought to discharge a duty implicitly laid upon Unesco by the Meeting of Experts who assembled in Paris in 1980 to consider the 'Goals of Mathematics Education'. One of their main conclusions was that the single biggest obstacle to progress in mathematics in most countries of the world was weakness in teacher education, and that assistance in this area of concern should be made available to Member States.

In contemplating the commissioning of contributors, it seemed to those responsible for planning the Volume that the issues fell into two broad categories: those which relate to the ever-changing curriculum and which bear heavily upon initial training, and those which relate more to providing day-to-day support for teachers in schools. The respective groups form Parts I and II of the Volume. The third part consists of two strikingly contrasting case studies: one describes in detail the evolution of the mathematics curriculum in China since the state came into being; the other describes a project initiated in Zimbabwe, the aim of which has been to design a learning 'package' for students in training which conveys insights into the various roles of a teacher of mathematics based upon the actual classroom practice of certain individual teachers whose success is widely recognized.

The contemporary trends in secondary school mathematics, which Part I elaborates are six in number. The first of these is the current concern for problem-solving. In this chapter, a number of thought-provoking questions are examined: what *is* problem-solving? what is a *problem*? in order to develop skill in solving problems, is it necessary to *start* with a problem? Discussion of these questions leads to the heuristics of problem-solving: questions which solvers should learn to ask themselves; strategies which may suggest a way forward; spotting 'cues' which may indicate fruitful avenues to explore; the place of personal reflection; the place of past experience; the burgeoning growth of the literature arising from studies made of mathematical problems and of problem-solving. Other issues examined include those which are specific to handling problem-solving sessions in the classroom: their frequency; the place of drill and practice; coping with frustration; the roles of comprehension, memory, intelligence and verbalization; defusing a problem by converting it into a situation; and the therapeutic value of posing problems, as opposed to solving them. And, by way of rounding out an unusually stimulating and thought-provoking piece of work, the Chapter concludes with a sequence of personal anecdotes illustrating the curious attitudes and unsuspected beliefs which some students hold about mathematical problems.

The second chapter is devoted to the influences upon mathematical education of environmental change. Various sources of influence are detected and sequentially examined: the emancipation of thought among students and teachers; evolutionary and revolutionary changes in primary school practice; social and cultural aspects. Contrasts are made between the implications of these issues for the more and the less developed countries of the world. And special attention is drawn to the importance of having regard to ethnic traditions within a society, particularly the effect these can have upon the mathematical performance of those who have had no formal schooling.

The third chapter examines the differences in mathematical achievement between boys and girls, and suggests actions which might be taken to reduce this gap. Comparisons are made of differences in attitudes about the ability and suitability of boys and girls studying mathematics. Research into this problem is presented, followed by actions taken by various groups to improve both the amount and quality of mathematics studied by girls.

The fourth topic of Part I is the impact upon the curriculum of the common availability of electronic calculators and the increasing availability of computers or of access to a computer. In the author's view, these resources are having three broad effects: they are influencing the existing content of the curriculum; they are introducing new content; and they are changing some of the methods of teaching. The most profound of these effects is the new meaning which the world 'numeracy' is acquiring. This, broadly, is now seen to mean the ability to use a calculator and to interpret its output in a manner which is consistent with the terms of the problem. It involves an ability to work mentally with single digit numbers, a facility with powers of 10, a grasp of place-value, and a recognition of the validity of results in relation to the accuracy of data. Other consequences are changes in the traditional order of teaching particular topics, new approaches to the teaching of algebra and new ways of imparting the concept of functionality. New content to be taught will obviously include algorithmic procedures, together with such ideas as iteration, recursion, error analysis, sorting, searching and modelling based upon real data. New methods will evidently include programming and the writing of special programmes designed to teach geometrical concepts, functionality and the calculus.

Chapter 5 examines trends in the traditional content of the curriculum: algebra, analysis and geometry. These are discussed in the context of the broad question: why and how does the mathematics curriculum change? Several factors are identified: response to changes in the subject itself (though classroom consequences can sometimes be counter-productive!); response to new applications of mathematics to, for example, traffic flow, environmental improvement, linear programming, graph theory, and Boolean algebra; response to the experience of failure of, for example, certain attempts made during the 1970s to present mathematics as a product of abstract deductive reasoning; response to changing views of what mathematical activity in schools should be like; and response to the availability of microcomputers and electronic calculators. The chapter concludes with a full but, necessarily, an inconclusive discussion of the place of geometry in the secondary school curriculum.

The sixth topic is statistics. Here, the place of statistics in the secondary curriculum is traced from the recommendations of the Organisation for Economic Co-operation and Development (OECD) conferences of the sixties in Athens, Cambridge, Royaumont and others. The nil effects of those in many countries is attributed to a lack of consensus on the foundations of statistical deduction, disagreement about the place of the subject in the curriculum, ignorance of how to teach statistics and a lack of qualified teachers. The case for statistics in secondary school education is re-examined, and its particular value is discussed in relation to model-building, simulation, susceptibility to the use of computers and calculators and to project work in schools. Finally, the implications for teacher education are examined and consequential proposals are put forward.

The first part of the Volume concludes with a detailed and comprehensive discussion of the place of mathematics within other subjects. Illustrative examples are taken from physics, biology, geography, astronomy, history, civics, economics, commerce, language study, arts and crafts and sport. The particular value of difference equations in solving problems of, for example, population growth and control, is pointed out, and many situations are described in which mathematical modelling has a particular place.

Part II of the Volume is devoted to providing support for classroom teachers. Three types of support are singled out for special study. These are in-school support, the rewards of membership of a professional association and the provision of in-service education. In the first context, a profound analysis is given in Chapter 8 of the role of the head of department and of the collective responsibility of members of the mathematics department for the welfare of mathematics teaching in the schools they serve. The day-to-day activities of the department are carefully analysed, and a helpful indication is given of how members can both support a new recruit as well as a member who has given long service. Suggestions based upon first-hand experience of good practice are offered on school-based training and curriculum development.

As to the role of associations of teachers of mathematics, three actual examples are taken and their respective activities examined: the Bolyai János Mathematikai Tarsulat of Hungary, the National Council of Teachers of Mathematics of the United States of America, and the Mathematical Association of Nigeria. Conclusions from these three are distilled to form a composite picture of the many ways in which membership in an association can confer benefit upon the individual teacher and upon the pupils.

The chapter on the in-service education of teachers of mathematics enunciates criteria for a well-designed course of training based upon experience gained at first-hand by the authors in the Arab States. These criteria relate to the context of the course, its content, its relevance to the needs of participants, the extent to which it requires individual participants to undertake private study, its clinical content and the extent to which it imparts understanding of the nature of mathematics itself. To illustrate the application of the criteria to particular cases, four typical models of in-service training are tested against them. As a commentary upon what is commonly on offer to teachers in-service, this contribution may well be deemed to be thought-provoking, practical and emimently valuable.

Part I

Contemporary trends in secondary school mathematics and their implications for teacher education

Stephen I. Brown

Problem-solving and teacher education: the humanism twixt models and muddles

Introduction

With all the confidence, pomp and circumstance accorded such causes as motherhood, apple pie, sushi, falafel and chicken soup (the latter for colds only), problem-solving is being ushered in as the paramount mathematics innovation of the eighties. A glance at the large number of talks devoted implicitly or explicitly to problem-solving at the International Congress of Mathematics Education (ICME) at Berkeley, California, is testimony to the trend. In its recommendations for new directions of the eighties, problem-solving is selected as the number one focus by the United States' major mathematics education organization, the National Council of Teachers of Mathematics (*An Agenda for Action: Recommendations for School Mathematics of the 1980's*, 1980). Entire yearbooks, chapters of research publication issues of national journals, and national and local conferences in mathematics education have been devoted to the subject (Krulik and Reys, 1980; Lester, 1980; Rachlin, 1982; and *Arithmetic Teacher*, November 1977, February 1982).

Why has problem-solving emerged as the focal point for curriculum and research? The question, in a way, is a foolish one. Instead of engaging in such heretical speculation, should we not rather thank the Gods that what was, or should have been, the case since the beginning of time, has finally been given its due? After all, what has the mathematics curriculum in the schools been about if not instruction in the solving of problems? Even the most pedestrian of teaching has engaged the student in solving problems for homework – despite the fact that this has been the bane of existence for many youngsters. Furthermore, research in the discipline of mathematics has progressed largely through the activity of problem-solving. The most famous example of progress in the discipline through an international effort to solve hitherto unsolved problems in mathematics is perhaps the one of David Hilbert. In 1900, he listed twenty-three unsolved problems, the solution of which he claimed would 'lift the veil behind which the future lies hidden' (Hilbert, 1900, p. 253).

To unpack what is behind the 'obviousness' of answers to questions which barely seem worth asking is perhaps the beginning of wisdom. To begin with, we might wish to inquire after those forces on the international scene that are creating the climate for a focus on research in problem-solving *at this time.* If problem-solving has always been an integral part of the mathematics curriculum, what are the political and social forces that contribute to the present climate for sounding the drums? And, as importantly, where are the pockets within the international scene for which the drums are silent, and what are the forces that account for that response?

Perhaps some recently-acquired 'hard evidence' to document our inadequacy in handling problem-solving in the schools has enabled us to see the disparity between pious hopes and achievement. A recent study of a representative sample of over 70,000 United States students of ages 9, 13 and 17 yielded the following conclusion about problem-solving:

> If it were necessary to single out one area that demands urgent attention, it would clearly be problem-solving. At all age levels and in virtually every content area, performance was extremely low on exercises requiring problem-solving [. . .] *In general, respondents demonstrated a lack of the most basic problem-solving skills* [emphasis theirs]. Rather than attempting to think through a problem and figure out what needed to be done to solve the problem, most respondents simply tried to apply a single arithmetic operation to the numbers in the problem. The results indicate that students are not familiar with such basic problem-solving strategies as drawing a picture of a figure described in a problem or checking the reasonableness of a result (Carpenter et al., 1980, p. 338).

But, from the perspective of a study in comparative education, possession of such 'hard evidence' is the veneer rather than the substance of the issue. That is, we are still left with the question of why we sought the evidence for the study at this time, and why we find lack of competence in problem-solving as devastating as we do. The issue of how much and what kind of thinking is desired or respected in a particular society is reflected not only in its inclination to create tests which include problem-solving, and in the manner by which that society reacts to the results, but also by the nature of the tasks it defines as problem-solving.

A major purpose of this chapter will be to attempt to locate some fundamental educational assumptions about the place of problem-solving in the mathematics curriculum and in teacher education. While it would be a significant contribution to knowledge to locate these assumptions within a social/political context, this paper will be less ambitious. By judiciously laying some groundwork of an educational nature, we may, however, provide some of the components that would contribute eventually towards a cross-cultural understanding that is rooted in a social/political context.

Our goal of locating fundamental educational assumptions will be accomplished not by carefully documenting what *is* the case as to the nature of problem-solving in the curriculum, and, concommitantly, in the preparation of teachers to deal with problem-solving in the classroom, but rather by speculating on how far from common practice we might venture in extending what appears to be standard paradigms both in the schools and in teacher education.

In taking on this job it is worth making explicit several assumptions that the reader may wish to challenge. First of all, we will be less concerned with locating differences among what some people would characterize as diverse programmes than in unearthing (and occasionally levelling or caricaturing) what they have in *common.* Second, in the realization that there are an infinite number of potential extensions, we have chosen a stance that might loosely be called 'humanistic'. We will be as much concerned with extending the concept of problem-solving along dimensions that enable us to question what it means to be a person (and to be educated) as with maintaining the integrity of the discipline of mathematics.

The activity of problem-solving

What is it that we call 'problem-solving' in the mathematics curriculum? One reason that so many people have found it easy to designate the eighties as the decade of 'problem-solving' is that the expression lends itself to such diverse and conflicting interpretations. As we have already indicated, according to one interpretation, 'teach problem-solving' is a relatively uncontroversial rallying cry because many people would agree that such activity has been the cornerstone of mathematics education for decades. Isn't it the case that one is solving problems when he or she is given fifty two-digit subtractions to do, after being taught how to subtract?

Other people would deny that this activity is problem-solving if the student, in some sense, already knows how to carry out the process of subtraction. It may take time to come up with an answer, and, in fact the student may make a mistake and come up with the wrong answer, but, if he or she knows before what to do to get the answer, then there is no problem. It is rather a task or an exercise. The belief here is that the desired product (the correct response) can be achieved through drill and practice alone. It is not that one has to memorize answers or even know them without working very hard, but rather that some already-known procedure guarantees success if it is followed correctly. This interpretation suggests that, given enough time, it would not be considered a problem for you to add two numbers with a million digits each (even in the absence of hand-held calculators) – assuming, of course, that no other physical or psychological problems were precipitated by the task.

How do we capture what it is that people are seeking who have another vision of problem-solving? The central concept that is generally advanced is that a problem should involve some sense of blockage, doubt or frustration.

This concept of problem-solving is one that has its roots in pragmatism, the philosophy of Charles S. Peirce and John Dewey. Nearly three-quarters of a century ago, Dewey (1910, p. 106) commented:

> The two limits of every unit of thinking are a perplexed, troubled, or confused situation at the beginning and a cleared-up, unified, resolved situation at the close.

For Dewey, problem-solving is not *a* mode of thinking reflectively; it is the *only* mode for such activity.[1] Borrowing largely from that same conception of a problem, many mathematics educators conceive of a problem as some goal that one attempts to achieve such that he/she is not aware of the procedure needed in order to achieve it at the time the problem is given.[2]

Among possible candidates for the category of 'a problem' in the school curriculum might be:

(1) 'Word problems' such as:

(a) A woman travels to work at 30 k/hr. for 2 hours. If she drives home along the same road at the rate of 40 k/hr., how long will the return trip take?
(b) A person has a collection of nickels (5c), dimes (10c), quarters (25c). If there are 25 coins all together, and there are 7 more nickels than dimes, how many coins of each kind are there if the total amount of money is $7.25?

(2) If one knows what it means to solve an equation, but has not yet learned how to solve cubic equations, the following might be problems in algebra:

(a) Solve for x: $x^3 - 3x^2 + 2x = 0$
(b) Solve for y: $y^3 - 6y^2 - 4y = 6$

(3) Problems or puzzles less obviously related to the content of the standard curriculum such as:

(a) Find all numbers x such that x^2 ends in x.
(b) Three people are sitting in a circle facing each other. A fourth one tells them that either one, two, or all three of them have a dark spot placed on their forehead. Except for observing each other, they are not permitted to communicate. Anyone who can prove

1. This footnote, together with all subsequent footnotes, can be found at the end of the chapter.

that he or she has a spot in the middle of his head wins a prize. Assuming that all three do in fact have spots (but none of the three is told that), can any of the three prove that he/she has a spot?

Whether or not all of these possible candidates will be accepted as problems depends upon the connotation of problem that one holds – beyond the first rough approximation of Dewey and of present day mathematics educators. There are people who would consider word problems to be the only legitimate candidates, and their major interest would be in helping students acquire the facility to handle problems of this type.

Others would deny that 'word problems' of type (1) (a) are legitimate problems at all if it is the case that the students have been taught a procedure for solving such problems (such as filling in boxes in a prescribed table, using a formula or drawing road maps). Most people, on the other hand, would admit that word problems of type (1) (b) are in fact legitimate problems (try it) on the grounds that standard algorithmic-type procedures (let x = the number of dimes, $7 + x$ = the number of nickels; $25 - (x + 7 + x)$ = the number of quarters, etc.) will not yield a solution in a way that would make sense to most students. In fact, there is an increased tendency in recent texts to include problems of this type – problems in which too much information is given, or not enough information is given, or in which redundancies appear.

While admitting that such problems as type (1) (b) may in fact be problems, there are an increasing number of people who believe that they are educationally dull – on the grounds that they lack real worldliness or realism. What is needed, some people claim (Lesh, 1981) are situations that are 'messy', and for which it is not clear what even counts as relevant information.

Whether or not one wishes to conceive of (2) (a) as a problem depends upon how much the student already knows about factorizing and solving quadratic equations by factorizing, as well as upon how one interprets the meaning of 'prior awareness' on the part of the student. Any student who has been taught to respond non-reflectively to an algebraic expression of the form given by first removing a common factor, and who has also been taught to draw immediately the implication that if $a. (b \cdot c) = 0$, then either $a = 0$ or $b = 0$ or $c = 0$, might very well not be thought of as having a problem, even though he/she is confronting something that appears to be novel. Example (2) (b), on the other hand, would most likely constitute a problem, even if the student has learned these 'lessons' well, provided he/she has not yet been taught how to solve a quadratic in general.

The two questions in (3) both raise another interesting issue regarding one's conception of what 'a problem' means. Since they are not part of the typical standard curriculum, it is conceivable that they will appear so foreign to the student that he/she may not even understand what is being asked. Though mathematics educators frequently contrast 'problem-solving' with the mere 'doing of exercises', they less frequently examine the other end of the spectrum; that is, to what extent can one be thought of as having a problem if the question being investigated is seen as chaos or nonsense?

None of the above analysis is intended to suggest that the present flurry of interest in problem-solving in the curriculum implies that educators believe that *only* problem-solving ought to be included. Surely most educators are happy to have some mixture of drill/practice and problem-solving as well. The difficulty, however, still remains as to what that part of the curriculum which is called 'problem-solving' should look like. This issue is open to considerable debate, and the reader may wish to decide which of the many possible components (including, perhaps, some we have not mentioned) of the concept of 'a problem' and of 'problem-solving' he/she wishes to hold on to.

A common pedagogical thread

Despite the diversity of opinion on what constitutes 'a problem' or 'problem-solving' (and there is much we have not addressed, some of which will be raised in the next two sections), there is a pedagogical unity that captures much of the diversity. It is that most articles, texts and teacher-education programmes that focus on problem-solving assume the following model:

$$\text{Problem} \rightarrow \text{Student} \rightarrow \text{Solve.}$$

That is, some source, such as a teacher or a text, gives a problem to a student, who is expected to try to solve the problem. Usually the problem is carefully defined beforehand, and the student is given a relatively short period of time to try to come up with the solution. How well-defined it is to be, and how short is 'short', are matters that require more careful exploration. But, as a rough approximation of the concept of 'short', it is worth noting that an approach considered by its founder to be radically different from other problem-solving programme boasts of problems that require 10 to 45 minutes for their solutions (Lesh, 1981).

There is considerable disagreement over the importance to be placed on the solution per se (an issue to be discussed in the section on teacher education), but it is clear in most of the contexts within which problem-solving is taken seriously that the solution of a problem is what provides the direction for investigation.

Now in some sense this view of the pedagogical intersection of different perspectives is rather pedestrian. That is, if one is interested in problem-solving, what other options are there? Here, it is important to appreciate that, despite the recent appreciation of the obviousness of problem-solving as a central component of mathematics education, we, as a profession, have not given careful enough consideration to such issues as:

- why we might want to start with a problem;
- what it means for something to be a problem;
- what it means for someone to be a student;
- the ways in which a student might relate to a problem;
- what it means to solve a problem.

Is there some sense in which we could be interested in problem-solving and yet *not* begin our problem-solving episodes with a problem *per se*? What we are implying by this collection of questions is that not only are each of the components of the intersection model in need of clarification, but, in addition, it is necessary to gain a clearer vision of what the 'near relatives' of a problem and problem-solving are, and how they might relate to problem-solving in the curriculum. These are issues we shall address in the section entitled, 'New Paradigms Needed.'

Problem solving and teacher education

If an explicit concern for problem-solving in the schools is a recent one, and if we do not have a consensus on fundamental issues such as what constitutes a problem and how the activity of problem-solving ought to proceed, we are that much more on virgin turf regarding policies for the education of teachers in the area of problem-solving. Whatever it is that does take place in formal pre-service teacher education programmes, does so in the context of each of the components of that programme. These consist usually of:

- courses in mathematics (usually comparable to what is required for an undergraduate major in mathematics);
- courses in mathematics education (dealing with instruction and curriculum);
- course work in the foundations of education;
- field experience, culminating in student teaching.

While reports exist which suggest the requirements for each of these components, and how they might be integrated (e.g. *Recommendations of Course Content for the Training of Teachers of Mathematics,* 1971; *Guidelines for the Preparation of Teachers of Mathematics,* 1973;

Overview and Analysis of School Mathematics: Grades K-12, 1975), it is difficult to determine how these components are orchestrated in practice. Case studies seem to indicate that there is an increased desire to integrate the components, and, especially, to take the field-based experiences more seriously than had heretofore been the case (e.g., Kerr and Lester, 1982; Wagner, 1982).[3]

Whether problem-solving is introduced as a unit in a course in the methods of teaching mathematics, or is a pervasive component of all college-level mathematics courses which students take, or is a source of occasional reflection in a methods course which 'shadows' a perspective teacher's mathematics course, would seem to make an important difference to how the activity of problem-solving is perceived.

Despite the fact that it may be difficult to determine the components of teacher-education programmes within which problem-solving is handled, it appears to be the case that, however it is orchestrated, a major assumption in those programmes that take problem-solving seriously is that the essential ingredient is for teachers to experience problem-solving in a manner similar to that which their youngsters might be expected to do. The assumptions here seem to be that, in order to be a good teacher of problem-solving, teachers need to become model problem-solvers themselves; and in order to become a good problem-solver, it is necessary to undergo considerable practice in the activity of problem-solving.

Towards the heuristics of problem-solving

The specific regimen varies a great deal, but a dominant theme appears to be that merely solving problems is not enough. In addition, some reflection upon and use of the *heuristics* of problem-solving is needed. Drawing upon formulations of Polya (1954, 1957, 1962), the central ingredient that distinguishes problem-solving (as we have described it so far) from Polya's concern is that of *process.* That is, to solve a problem (or to attempt to solve a problem) is not merely to come up with an answer or a solution, but it is to engage in a thought process as well. Questions like the following are the hallmark of heuristic thinking:

- What is the problem really asking?
- How can I get started?
- What should I do when I get stuck?
- What have I really found out in this problem?

Anyone, for example, who is confused by a negative fractional value for the number of dimes as a solution to problem (1) (b) would gain more enlightenment by asking questions like:

- Is the problem a reasonable one?

- What are the upper and lower bounds that I might expect as an answer, given the initial information?
- If all my change were quarters, how much money would I have all together?

than by re-doing the algebra and attempting to search for an error in calculation.[4]

Anyone who is confused as to how to begin to think about problem (2) (b), might profit from making a sketch of the graph of the function $f(y) = y^3 - 6y^2 - 4y - 6$. If there is no known procedure to answer the question precisely, then successive approximations to the correct answer may be achieved by judicious guesses that hover around values for which f(y) comes close to zero.

Similarly, one may gain some understanding of problem (3) (b) by looking at specific cases – merely plugging in values of x. If $x = 2$, then $x^2 = 4$; if $x = 4$, then $x^2 = 16$. Neither of these conditions satisfy the problem. If $x = 5$ then $x^2 = 25$. Here we have something that 'smells like' it might work. Do I now have a better understanding of what is being asked for in this problem? Let me try some more numbers greater than 5. What happens if $x = 25$ (assuming some recursive stance)?

Teachers can be introduced to the heuristics of problem-solving and be prepared to teach these heuristics in a variety of ways. A popular model appears to be one in which teachers are given problems to solve that are at their own level of sophistication rather than at the level of the students they will teach. When blocked in attempting to solve problems that they have not previously confronted, they are encouraged to explicate strategies they either are using or might use for the purpose of trying to solve the problem.

In the process of developing a model for themselves, they are persuaded to make use of similar strategies in their own teaching. They might be taught to make use of these strategies explicitly in their own teaching. Thus, in finding their own students 'stuck' with a problem, prospective teachers might be encouraged to prompt the students to ask themselves explicitly:

- What do I know that is like this problem?
- Do I see a pattern?
- Can I guess what an answer might be?
- Can I approximate an answer?

And when a solution to a problem has been found, they might be taught to encourage their students to ask questions like:

- Are there other approaches I might have taken?

- What have I really done?
- What else might I now explore?

Alternatively, they could be taught to make use of these heuristics implicitly in their own teaching. Instead of prompting their students explicitly to ask questions of the type suggested above when they get 'stuck', teachers could be taught to ask similar questions themselves. Thus, the teachers might question their students thus:

- What do you know?
- Can you think of a similar problem?
- Can you think of a special case?

The implication here is that students might learn to internalize the heuristics of problem-solving without necessarily being taught them explicitly.

A fine point in heuristics

There are surely a large number of problems in the universe as well as a large number of heuristics available to solve them. The task of the problem-solver is not merely to make use of heuristics when 'stuck', but rather to make use of reasonable or appropriate heuristics. The matter is even deeper, however, because any one of the heuristics has an ambiguity that lends itself to many different interpretations, and it makes a significant difference which interpretation of the heuristic is used in a particular problem. Though there appears to be very little research done on how one might teach students to apply reasonable heuristics to particular problems, Schoenfeld (1980) has provided the beginnings of a conceptual analysis which would enable teachers to find appropriate cues in particular problems that might signal which heuristics to use, and which of the many different interpretations of a particular heuristic might be appropriate.

Consider problem (3) (b) dealing with the people having spots in the middle of their foreheads. Speculation might run along these lines:

Is there anything in the content or the structure of the problem that suggests some strategy to begin an analysis? Suppose there are three people, A, B, C. If I am person A, suppose that I notice that B and C do have spots, but that neither has indicated that he/she believes that he/she has a spot. Why are they not raising their hands? I do not know whether or not I have a spot. What should I do? How do I reconcile the fact that B and C are not indicating whether or not they have spots with my condition? There are two possibilities. Either I have a spot in the middle of my forehead or I do not. That is, there either exists a spot or there does not exist one. This last phrasing of the problem perhaps suggests a heuristic to use. Suppose I did not have a spot in the middle of my head. Then what would follow?

Without spoiling the analysis for you, let me suggest that a not too tortuous argument would lead me to the conclusion that either B or C would have raised their hands. But neither one has done so. The final conclusion now is obvious.

Now this particular problem may not be the best one to illustrate how the problem itself contains a clue that a proof by contradiction might appropriate, for considerable analysis was required beyond the content or structure of the problem itself in order to reveal the heuristic. Shoenfeld has pointed out that such clues as negative conclusions, or conclusions that are not apparent, or that use 'unique' in the conclusion might signal the occasion for reasoning by contradiction. In this case, I have suggested that viewing the problem from the perspective of *existence* might also be a cue.

A blunt point on heuristics

It is worth pointing out that Schoenfeld's analysis of the relationship between heuristics and specific problems depends upon conceptual analysis resulting from personal reflection and analysis of one's own experience in doing mathematics rather upon research of a strictly empirical nature. Before making light of such research, however, one ought to recall the nature of the research which spawned not only our concern with the heuristics of problem-solving, but with empirical research in the area as well – namely, that of Polya. The fact that Polya's conceptual analysis and introspection (rather than 'hard evidence') enabled him to make sense out of the world for others could perhaps be a source of inspiration for teachers who are searching for ways of handling problem-solving in their own classrooms.

Most empirical work grows out of a conceptualization of a theoretical nature (whether it is articulated as such or not). Furthermore, it is frequently the case that the *categories* that are generated by theory are themselves what is most valuable for teachers – rather than empirical findings. Though our focus in this section has been on *heuristics* and problem-solving, there are surely a host of other such categories that are important and relevant for teachers who are interested in incorporating problem-solving in their classrooms.

Issues like:

- the timing of problem-solving in the classroom,
- its relationship to drill and practice,
- its relationship to the acquisition of knowledge,
- the level of frustration desired during the activity,

– the impact of the structure of the problem on students ability to comprehend,
– the place of memory in problem-solving,
– the relationship between intelligence and problem-solving ability, and
– the role of verbalizing in problem-solving,

are surely one that teachers will confront.

Some of these issues represent categories within which research in mathematics education has been conducted (see Lester, 1980, Kantowski, 1980, for recent summaries). As suggested above, once they were made aware of some of these categories, teachers and teacher educators may find it as helpful to analysis them conceptually and to reflect on their own experience as to search for relevant findings.

In particular, it may be helpful to begin by assuming a healthy dose of skepticism and distrust of dualisms that are established between problem-solving and related categories. At a recent conference, for example, a colleague of mine suggested that since problem-solving does not take place in a vacuum, it is necessary to spend considerable time establishing knowledge so that one might have what is a necessary prerequisite to solve problems.

Now this suggested bifurcation between knowledge-acquisition and problem-solving begins with a grain of truth and ultimately draws implications that are highly questionable. The truth is that problem-solving, like critical thinking, is in fact a binary relationship. One would wonder about the sanity of a person who responded to one question 'what problem are you solving' with, 'Oh, no problem in particular. I'm just solving problems in general'. One solves particular problems or clusters of problems. A central question then is: How does one acquire the knowledge that will be used as the substance for later problem-solving? The implication of my colleague is that this is done in the absence of problem-solving – as though filling an empty vessel was a prerequisite activity.

Now this is an issue which needs a lot more thinking through, even before any empirical research can be done. Though some of what we say in the next section may provide further stimulus for personal reflection, the reader might make some headway in thinking about this problem by trying to explain how it is that he/she gained competence over any body of knowledge of which he/she feels in control. To what extent was the body of knowledge acquired in the total absence of problem-solving strategies? Were there any elements of problem-solving strategies that were used in the activity of acquiring competence?[5]

New paradigms needed

Though the state of the art has progressed somewhat since Begle's (1979) summary of the value of research findings on problem solving, it is sobering to recall the conclusion he reached based upon a review of empirical evidence (on the topic up to 1976):

> This brief review of what we know about mathematical problem-solving is rather discouraging. Compared to the importance of the topic, the amount of factual information that is available to us is quite small. Even more discouraging is that little interesting research is going on at present.

There is no doubt that more interesting research (especially explorations of a non-statistical, case-study variety) has been going on in the short time since Begle's summary. Nevertheless, the field is in its infancy, and the problem is not merely that the evidence has not come in, but, more importantly, we do not have an adequate view of the different paradigms within which inquiry might be conducted.

In this section, I would like to point towards alternative world views for such inquiry. In doing so, I will not be making sharp distinctions between problem-solving activity in the mathematics classroom of the schools and that of teachers education. Our vision of both these dimensions is surely clouded. And, though I will point towards alternative teacher-education programme, it is important to appreciate that we have not yet had sufficiently adequate dialogue to know what it is we ought to be doing to prepare teachers for problem-solving in the classroom. Indeed, it is conceivable that clearer perceptions of teacher education might even redirect (or revolutionize?) what takes place in classrooms, rather than prepare teachers to handle adequately what is perhaps not worth taking seriously to begin with. In attempting to expand the paradigm along dimensions that I perceive as humanistic, I will attempt to pinpoint several fallacies. Despite my effort to separate them for ease of exposition, they are clearly intertwined. Furthermore, in most cases, any implied embeddedness of the categories could be easily reversed, depending upon slight amplification or redirection of the lenses through which these fallacies are viewed. We leave it as an exercise for the reader to provide an intertwining and embeddedness of suggested categories as a function of his/her understanding of what is educationally important and most in need of repair.

The fallacy of 'the given'

A large part of one's mathematics education derives from a world view in which we 'assume the given'. Indeed, proving things in mathematics, for the most part, requires that we demonstrate that p implies q, and that demonstration for the most part is neutral about the truth of p. That is, we assume p to be true, and try to demonstrate q. We are not concerned with the question whether p is in fact true or not. Now, clearly, different philosophies of mathematics handle the question of what is truth and how it relates to mathematics differently. But such delicate theoretical issues are, for the most part, suppressed in school practice. The central issue with education, however, is that 'the given' is taken as the thing to be *accepted*, and the one from which implications are to be drawn.

The taking of 'accepting the given' as our point of departure has had an enormous impact on what we perceive problem-solving in the curriculum to be. What is it that we as teachers give our students when our interest is in problem-solving? As suggested in the section 'A Common Pedagogical Thread' (pp. 8–9), we give them a problem.

What is the job of students? It is obviously to accept the problem and to work on it. Now what it is that people accept and how they accept it is a matter that deserves careful attention. If you give me a problem in a non-school setting, there are all kinds of options open to me. I can accept the problem and try to solve it. I can ignore the problem. I can, on the other hand, take the problem and do something worthwhile with it that is neither of these activities.

What are some of the things I might do with a problem short of ignoring it or trying to solve it? Just to ask the question is to suggest that we have become stuck in a paradigm that might be worth challenging. Let us return to problem (1) (b) – the coin problem. A reasonable first approximation might be for me to ask you why you have given me the problem to begin with. After all, there are all kinds of motivations teachers have for giving problems to students. Does one have a right to know the context within which problems are given? For example, might it be worth my while knowing whether you are giving the problem to me:

- to improve my skills in some domain?
- to keep me busy?
- to lead me to totally new conceptualizations of what a problem is?
- to prepare me to solve others like it for an exam that you will give me?

The fervour and the time I devote to most tasks is usually related to my purpose in taking on the task. Why do we not feel as comfortable in revealing our educational purposes? None of this is to imply that a student should be in the position to distrust a teacher. First of all, we engage in such inquiry when our closest friends ask us to participate in their lives. Second, it may very well be the case that teachers who are unaware of why they submit students to particular tasks might profit from a dialogue in which some clarity might emerge.

But requesting dialogue on purposes is just a start towards seeing what our response might be to a problem we are given other than to solve it or reject it.

Another possibility might be to try to 'de-problem' the problem. That is to say, instead of trying to solve it, I might reconstruct it as a *situation* that encompasses more than the problem I have been given. Given the coin problem, how might I re-create it as a situation? One possibility might be to offer the statement:

A person has a collection of nickels, dimes and quarters. There are 25 coins all together and seven more nickels (5c) than dimes (10c). The total amount of money is $7.25.

Is this now a *situation* for you rather than a problem? If it is still a problem, why is it so, and what would be needed to neutralize it?

It would seem to be an invaluable educational tool for teachers to inquire when it is that their students see and can accept a situation that is relatively neutral and when they see a *problem* that appears to call for some action on someone's part. Furthermore, we could all learn a great deal about the society within which we live and about other cultures as well by coming to understand what are the problems which appear incapable of being 'de-problemed'.

Attempts to 'prove' Euclid's fifth postulate, for example, tells us a great deal about the nature of mathematical thinking for a period of over 2000 years. Because it appeared to be less fundamental than the other postulates of Euclid, people asked the following question for a very long period of time:

How can you prove the parallel postulate (There is exactly one line parallel to a given line through an external point) from the other postulates of Euclid?

The fervor with which the question was asked and analysed covers much mathematical ground – culminating in non-Euclidean geometry, and, as importantly, a totally new conception of what a system of axioms is all about. Along the way, however, we find out something about the kinds of convictions people hold. In fact, the entire mathematical world was incapable of neutralizing that problem long enough to see it as a situation.

Once we have defused a problem by converting it into a situation, what is it that we might do of an intellectually valuable nature with the result? Obviously, we cannot 'solve' a situation, as we can a problem. Yet, why not? A situation does not in and of itself ask a question, though we might impose questions or problems on a situation.

We are thus led to an insight that has taken a number of years for me to appreciate and to begin to unravel. Students can be placed in a position to *pose* problems, not merely to *solve* them. Problem-posing (or generating) is as intellectually respectable an enterprise as problem-solving, and, though the two activities are intertwined in many ways, they have threads that are separable.

They are intertwined in the sense that a poor, or inappropriate, posing of a problem frequently prevents us from solving it (and the attempt to 'prove' the parallel postulate is an excellent example). They are also intertwined in that it is necessary to repose any problem in order to solve it; and it is frequently the case that we cannot understand an alleged solution to a problem until we pose other questions about it. Consider the following example:

If you have two similiar equilateral triangles with sides a and b, find the length c, of a third equilateral and similar triangle, such that its area equals the sum of the other two.

Some straightforward calculation reveals that $c^2 = a^2 + b^2$. Now anyone coming upon that conclusion by mere calculation would be puzzled by the unexpected connection between this problem and the Pythagorean theorem. In order to make sense of the alleged solution, some new set of problems must be generated – beginning with something like: 'What is there about this problem (that deals with similar equilateral triangles rather than squares) that is reducing it to a problem involving squares?'[6]

But problem-posing or problem-generating can also be appreciated in isolation from problem-solving. We have suggested already that the existence of a situation might be an opportunity to pose a problem. But problems themselves can provide the opportunity not for accepting them and solving them, but for generating totally new problems as well – problems that are not generated for the purpose of extending or shedding light on the original one.

Though Marion Walter and I have begun to establish the logical terrain within which problem-generating might reside, and have, in addition, suggested strategies for generating problems and teaching others to do so, issues of how such a programme might be incorporated into the curriculum, and what the related research questions might be, are in need of considerably more wisdom than we have provided.[7] A significant educational fall-out of this line of inquiry, however, may be the discovery that the talents needed to generate and to solve problems are not as closely linked as we might imagine. The opportunity for students with these different talents to collaborate suggests new instructional possibilities as well as research potential.

The fallacy of the context of the problem

Though clearly the field of mathematics education depends upon the discipline of mathematics for part of its definition, it is not at all clear what the nature of that dependence ought to be. After all, education is a broader and more problematic a construct than mathematics itself, and while there are many calls for integrating the disciplines (e.g., as in applications of mathematics to other fields) and in integrating the components of teacher training (such as combining courses in methodology and the discipline of mathematics), much more is borrowed not only from the content but from our beliefs about the nature of mathematics than we generally acknowledge.

What are these beliefs?

First of all, there is the world view that asserts that in mathematics the establishment of truths is relatively easy, surely in comparison with fields that require worldly experience, like philosophy or literature. This is not to deny that there exist unsolved problems, nor to imply that most people would not have difficulty in understanding higher level mathematics. Rather, it is to suggest that an appropriate level of intelligence, coupled with a well-developed sequence of mathematical experiences (the latter being harder to come by than the former), will enable one to understand and to solve problems with considerably more ease than is the case in the humanities.

Second, there is the world view that mathematics searches for generality and abstraction. Certainly, for pedagogical purposes, a great deal of mathematical experience is 'concrete' in nature. But there appears to be an assumption that such approaches to the curriculum are a function of the immaturity and lack of intelligence of students. If the clientèle were only clever enough, we could move rather quickly to the demonstration of isomorphic structures, and thus show the underlying unity of ostensibly different structures.

Third, there is a general assumption that personality does not enter into mathematical investigation. The basic assumption here is that, since logical analysis is the final arbiter of what is correct and what is not, the personality, the culture, the character of the investigator is irrelevant from the point of view of research into the discipline. Though the activities of teaching and learning tend to soften the blow of this message (and in fact good teachers surely take into consideration the feelings and insecurities of their students) the general fall-out from that perspective is maintained. That is to say, one's major goal in teaching is to strive to impart understanding of mathematics and awarenesses that are of the discipline and not of the self.

The point here is that, not only is the *content* of mathematics education driven by the discipline of mathematics, but that the world view of the discipline is what is latched onto in educational settings as well. It is interesting that all of these views (and others, which the reader might wish to generate) are maintained, despite the fact that there has emerged over the past few years an appreciation of mathematics as something *less certain, less easily verified, more personal than is generally acknowledged* (Davis and Hersh (1981); Hofstadter (1979); Klein (1980); Lakatos (1977); Polanyi (1962)).

The world view of mathematics clearly influences problem-solving as well. It is a perspective on the notion of 'problem' that is well-captured in the following remark by Sarason (1978, p. 374) on scientific problem-solving:

> A problem has been 'solved' (a) when it does not have to be solved again because the operations that lead to the solution can be demonstrated to be independent of who performs them, (b) when the solution is an answer to a question or set of related questions, and (c) when there is no longer any doubt that the answer is the correct one.

Sarason appreciates that in scientific or mathematical inquiry one can make mistakes, and it is possible, for a while, to have competing or incompatible solutions, but the implication is that these issues will be settled in time. He correctly points out that problems in the humanities are of a different order of magnitude. Using social history as a focus of comparison, he points out that the concept of 'problem' is of a different order of magnitude. One does not confront the problem of loneliness or ultimate death on Monday, solve it and move on to how to justify living on Tuesday. Problems in the humanities are more like re-current themes over which one may gain greater insight, but one surely does not solve these problems as one does in mathematics.

If we take our paradigm from the humanities, and apply it to problem-solving in mathematics education (not forgetting that education is a broader and more problematic field than is mathematics), then possibilities emerge that we have barely begun to imagine.

First of all, our time frame for doing problem-solving might be expanded by even more than the extravagance of 10 to 45 minutes desired by Lesh (1981). Instead of selecting problems that are relatively easy, and for which we might expect a solution in a reasonable period of time, we might select some problems from among paradoxes or from among the unsolved problems of mathematics. Many of the unsolved problems are easy to state (e.g., Goldbach's conjecture, Fermat's last theorem, the four-colour problem – assuming the computer solution is viewed as lacking enough in aesthetic appeal to warrant our continuing to view it as an unsolved problem).

While it would be more than unrealistic for us to expect that relatively inexperienced students would be able to solve any of these problems, we would surely push the bounds of what we mean when we claim that 'process takes precedence over product' by so engaging students. The suggestion here is not that students report on these problems by citing popularized research literature, but rather that they actually engage in some of the process activities that the profession has called for in recent years, as well as some of what we have advocated in the prior subsection. That is, instead of expecting answers, we might expect partial solutions, approximate rather than exact answers, attempts at solving simpler problems and even generating new problems based upon the starting point of the unsolved problems.[8]

But the issue of viewing 'problem' from the perspective of the humanities rather from that of science alone opens up new options beyond that of the selection of problems of mathematics. Since mathematics education can be viewed as a field that uses mathematics for purposes that are educational, there are totally new conceptions available. Mathematics can be used, not only to enhance one's mathematical competence, but for purposes of reflection of a personal nature that may not have wide applicability to the population at large.

Several years ago, I was teaching students to view the same problem from many different perspectives. I selected the famous Gauss problem: Find the sum of the numbers from 1 to 100. In addition to using the string of numbers, I encouraged them to use the staircase as a model as well. They created paper models, and folded and cut their models. After several days, the students had generated many different forms for the answer to the question. Among them were:

(i) $\frac{n}{2} \cdot (n+1)$

(ii) $1/2n^2 + 1/2n$

(iii) $n \cdot \frac{(n+1)}{2}$

(iv) $\frac{n \cdot (n+1)}{2}$

(v) $(n+1) \cdot \frac{(n-1)}{2} + \frac{(n+1)}{2}$

(vi) $1/2\,(n+1)^2 - 1/2\,(n+1)$

In hoping to get them to reflect upon the heuristics of problem-solving, I asked them to say which of the many answers generated they preferred. I was surprised to find that some students selected what appeared to me to be the most complicated (e.g., (v) and (vi)). Subsequent discussion revealed that some people felt the need to maintain some sense of the evolution of an idea – the history of its derivation – along with its solution. They were essentially saying something about *what it* *means to be a person*, and, unlike what we might expect from a computer, they expressed the need to know how their present 'selves' related to what they were a while back. Here, surely, is a theme that one might spend a considerable time analysing. In fact, the more general issue of what it means to be a person in relationship to one's history is one that could profitably be explored and re-explored over a lifetime. What a fundamental error it is to divorce an understanding of oneself from the study of a discipline.

The fallacy of the empty vessel

In studying phenomena like mathematical anxiety, we are beginning to relearn more generally the important educational lesson we once learned through the findings of Piaget and of developmental psychology. It is that, regardless of the age and level of sophistication of our students, they have attitudes, values, beliefs and feelings which form a window that affects how they see the world.

It is perhaps worth our taking stock every so often to try to understand the kinds of problems they think they are solving, even when they are at the most advanced cognitive stages of development. This is a lesson I find myself relearning at regular intervals – and when I least expect it. Two years ago, for example, I remember teaching my son, Jordan, how to handle the ambiguous case in trigonometry (the conditions under which two triangles are congruent if they correspond with regard to angle, side, side)[9] He seemed confused. So I backtracked and asked him to recall that two years earlier he had learned something about the conditions that determine a triangle. His response was: 'No, I did not learn that. I learned how to prove two triangles congruent by S.A.S., A.S.A., and S.S.S. His belief-structure was completely at odds with what mine would have been. The wonderful irony here is that, while I saw the conditions of congruence as providing the various sets of data which determine a triangle, his more limited and alternative view (spawned by a textbook author to provide experience in proving theorems) enabled him to answer correctly all questions put to him, despite the fact that he had no view of what I perceived to be the grand scheme.[10] I have been similarly 'caught' many such times by students taking courses in teacher education, or in interviews I have had with teachers. I set out here several poignant incidents, and would like to use them as a stepping stone for reflection on teacher education.

(1) The Story of Sam

As a prerequisite for developing the complex numbers carefully, I gave my students the following problem: In grade 9, you proved that $x^2 = -1$ has no solution. In grade 11, you learn that if you assume everything you assumed in grade 9, but add the stipulation

that $x^2 = -1$ has a solution ($\sqrt{-1} = i$), then you can create a new number system called 'the complex numbers'. How can this be? One of the students, Sam, responded:
'That's not surprising. You frequently learn when you get older that what you thought was so, was wrong.'

(2) The Story of Robin

Robin was complaining that in her student teaching experience, none of the students were intellectually interested in anything. The only thing that motivated them to do any work was the prospect of getting a grade. In an effort to understand better what she was saying, I asked her (a Phi Beta Kappa mathematics major) what were some things that motivated her in school. The response was an uncomfortable, embarrassing silence. She was on the verge of tears.

(3) The Story of Marsha

As part of a research project on values held by teachers, I asked Marsha what learning mathematics was like for her. She was asked to close her eyes and respond as if she were back in school. Her review was magnificent. She spoke of the flowers; she spoke of education as climbing mountains with a guide. The tone was sensual, stimulating and, according to her own language, 'very romantic.' At another point in our interview, I asked her what she does with students in her own teaching (especially if the students have difficulty). She responded that she makes them learn every detail, and to say every word when they write each symbol.[11]

These three stories clearly tell us something about the value of listening carefully, as well as about the power of the beliefs people hold. What is interesting about story (1) is that it revealed an attitude towards mathematics that never would have emerged if I had asked: 'Do you believe mathematics is an inconsistent discipline?' It was in the context of supposedly talking about a specific isolated problem that the more general 'bomb-shell' of a belief emerged – that mathematics might hold the propositions 'x' and 'not x' simultaneously. The power of listening for 'the general' as one talks about 'the specific' is one that we might better learn to harness in our educational dialogue with teachers.

The second story reveals yet another turn of events. A prospective *teacher* has been a *student* all her life. Many of the beliefs and values held in the one sphere have not been well integrated into the other. In learning to talk like a teacher, Robin has not yet learned how that language impinges on her own experience as a student. The student within her is, in fact, a very powerful potential resource against which she might set the beliefs of the emerging teacher.

The same point is revealed in the third story, though Marsha has not yet begun to understand the inconsistency of behaviour and attitude brought out by the student and the teacher within her.

All three stories point in the direction of a new set of metaphors for teachers education – metaphors that are less directed towards the 'input' into a supposedly empty vessel, and more consciously concerned with what is already there. The perspective we derive from developmental psychology seems to be that a student's world view is worth knowing, so that we can then fill him/her up so as to dilute adulteration. It is perhaps worth borrowing our metaphors more from therapy than from developmental psychology.

It is not that we, as teacher educators, need to perceive our role to be primarily one of 'filling up the vessel' once it is 'cleaned out'. The metaphor of therapy suggests that once teachers are given the opportunity to hear what students do in fact believe – something that may be painstakingly long to hear – the students themselves might be in a position to modify significantly what they find wanting.

Creating a climate for listening in the context of mathematics education is clearly not a passive activity on the part of the teacher-educator or the prospective teacher. As the three anecdotes suggest, it is the existence of a 'rich' environment which provides the backdrop against which teachers learn to reveal to themselves what they actually believe, what constitutes 'rich', and how each teacher-educator creates it, is a fundamental problem (in the humanistic sense of 'problem') that we each must confront.

In my own work in teacher education, I have found a powerful device for incorporating the twin concerns of teaching about problem-solving and problem-posing and encouraging students to unearth fundamental beliefs about teaching, learning and mathematics. The device is to employ the metaphor of teacher as an editorial board member and as an author. In several of my courses, in-service teachers are divided into several competing editorial boards. As we begin to explore strategies for both posing and solving problems, participants write articles about pedagogical or mathematical problems that either they or I have defined. They are not expected necessarily to solve these problems, but, rather, it is intended that they will mull them over; record their emotional, logical analysis; indicate new issues that come up in the process; find ways of talking about and accounting for moments of insight as well as for abortive efforts; and so forth.

Each student writes papers (sometimes in collaboration with others) to be submitted to an editorial board of which he or she is not a member, and each student participates in the analysis and judgement of papers submitted to the board of which he or she is a member.

With the passage of time and experience, each editorial board formally establishes a policy and advertises it so that colleagues can decide to which board they wish to submit papers. The boards decide to accept, reject or require revisions of papers submitted, and at the end of the semester each board produces an actual journal consisting of the accepted papers and letters of acceptance (indicating why they liked the paper), as well as any other self-defined activity (such as a contest, or a collection of annotated readings).[12]

Though I have found this model useful for getting at much of what I believe is needed in the way of new paradigms, the model is not without its difficulties. Furthermore, each teacher-educator will have to find an environment that he/she considers conducive to the goals of his/her own emerging paradigm.

Why teach problem-solving?

We seemed to have strayed far from problem-solving in mathematics education as we conclude with a vague and ambiguous conception of 'therapy' as a model for teacher education. If we appear to have left the field of problem-solving altogether, it is perhaps a result of a final fallacy that I would like to suggest. It is that, if problem-solving is viewed as a worthwhile activity in the mathematics education of students, what is the analogy for teacher education? The task of this chapter originally assumed that the answer is: 'How to teach problem-solving'. That is to say that the following analogy was implied:

Students:	Problem-Solving: in mathematics	Teachers:	Teacher Problem-Solving
(a)	(b)	(c)	(d)

I have suggested not only that (b) is in need of reconceptualization by taking the 'education' component of 'mathematics education' more seriously, but that (d) might more appropriately be replaced with a view of teaching as a problematic enterprise.

In order to appreciate how it is that one might arrive at such an extension, let us return to a question that inspired this paper: Why teach problem-solving? Though we have not begun to answer the question along social/political lines, an educational response is embedded in this chapter. It is that the importance of problem-solving is not in providing the wherewithal to solve better as many new problems as possible, but rather to learn to survive in a problematic world for which both the questions and the answers are uncertain. If coping with uncertainty for students means operating in an atmosphere which reduces algorithmic thinking, and which places a premium on processes in which the concept of failure is deemed inappropriate, then what are the analogous risks and uncertainties for teachers?

Given where we have ended up, and given the diversity and richness of the concept of 'problem', perhaps the nicest way of concluding this chapter is to suggest that the most important reason for teaching problem-solving is to provide the atmosphere within which we are all (students, teachers, parents, administrators and teacher educators) encouraged to ask the question: Why teach problem-solving?

Footnotes

1. See Scheffler (1973, chapter 12) for an excellent discussion of the shortcomings of a conception of thinking which reduces it to problem-solving.
2. Our purpose here is not to offer a philosophically accurate conception of what a problem is and how problem-solving relates to the conception of a problem. Rather we are attempting at this point to reflect how the notion of problem-solving is used in the profession. There are a number of philosophical problems that are related to this loose conception and are worth investigating in their own right. For example, does it make sense to speak of a person having a problem that he/she is unaware of? To what extent does the concept of problem depend upon the concept of solution? These and other such matters are discussed in the excellent essay by Gene P. Agre (1982). We will raise some of these kinds of issues in an educational context in the section entitled 'New Paradigms Needed' (pp. 15).
3. The recent turnabout from over abundance to teacher-shortage in school mathematics in the United States will surely have an impact on the manner and speed by which teachers will be certified.
4. None of what is said about the use of heuristics is meant to suggest that anyone who believes he has made an error in algebra is foolish. Especially if most problems of this type are, in fact, 'reasonable' problems – unlike this one. If this is one's experience, it makes good sense for one to so explore the conundrum. For a more detailed discussion of this problem, see Brown (1981).
5. See Brown and Rising (1978) for further discussion of the power of conceptual analysis as an alternative to empirical study of key educational concepts in mathematics education.
6. See Walter and Brown (1977) for an elaboration of this particular problem.
7. See the following pieces for an analysis of parts of the terrain of problems generating: Brown (1976); Brown (1981); Brown and Walter (1970, 1983); Walter and Brown (1969, 1971).
8. See Brown (1973, 1976, 1982) for a discussion of other related themes of this sort.
9. This matter is discussed in a slightly different context in Brown and Walter (1983).
10. A similar point is well documented in Erlwanger (1973).
11. See Brown, Catherine, et. al., (1982) for a discussion of the project within which this research was conducted.
12. See appendix of Brown and Walter (1983) for a discussion of the details of this model.

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Ubiratan D'Ambrosio

Environmental influences

Introduction

In this chapter we shall discuss several aspects of the inter-relationship of mathematics education with the environment, more particularly in relation to the secondary level, and with special reference to their implications for teacher training.

We begin, however, with some general considerations of the changing personality of students and of teachers, their expectations and their setbacks, and how these are related to mathematics education. We go on to discuss school systems. We shall refer especially to the profound changes in the primary schools as they move into the electronic era, and the obvious consequences for secondary education that these changes will have very soon. This will impinge directly upon teacher training. A new curriculum for the primary school is discussed, together with its implications for secondary students' behaviour. We then consider the environment in its broadest sense. Its social and cultural aspects are emphasized, and a lengthy discussion is given of the consequential peculiarities of mathematics education in the countries of the third world. Finally, there is a discussion of environmental factors, as these affect mathematics education.

Up to this point, the paper is a development of certain considerations which originated in a study prepared for the Third International Congress on Mathematical Education (D'Ambrosio, 1976), and which was published later in a shorter version by Unesco (D'Ambrosio, 1979a). By way of conclusion, reference is made to some other studies and projects which are relevant to the theme.

The students

To give the theme a focus, we begin with a discussion of students' perspectives in general, with particular reference to those of the adolescent. To do so, we must remind ourselves who the students are who go to secondary school. First, their ages. This, in most countries of the world, will run from about 14 or 15 to about 18 or 20.

They represent a somewhat privileged group in society. Yet, among themselves, they do not always adhere to the conventions of the social group from which they come. This means a growing level of political involvement, of social concern and of commitment to change. On the other hand, conflict with the expectations of their parents, incurs strong reaction to change.

In 1906, the Austrian, Robert Musil, one of the most important writers in the German language, published one of the masterpieces of the literature of this country: *Die Verwirrungen des Zöglings Törless* (The Confusions of the Young Törless). It is a study of adolescence seen through the eyes of a sixteen-year-old student at a selective military academy. Among the various experiences of the young Törless is a discussion about imaginary numbers with his fellow student Beineberg, and a subsequent interview with his professor of mathematics.

The dialogues quoted reflect a commonly held external view of what mathematics is and how mathematics in school often proceeds among students in the age bracket with whom we are concerned in this paper. The views which people who are outside the circle of practitioners of mathematics – and here I place mathematicians as well as teachers – hold about the discipline, as, for example, those of Robert Musil's characters, reflect in a very clear way an image typically held by the general public and so by our 'customers', the students. Whereas we try in the school system to impart an idea, or to convey a message, the preconceptions of the student frequently determine, to a large extent, the success or failure of our efforts. Here, first is the student dialogue:

During the mathematics period Törless was suddenly struck by an idea . . . .

Törless – 'I say, did you really understand all that stuff?'

Beineberg – 'What stuff?'

Törless – 'All that about imaginary numbers.'

Beineberg – 'Yes. It's not particularly difficult, is it? All you have to do is remember that the square root of minus one is the basic unit you work with'.

Törless – 'But that's just it. I mean, there's no such thing. The square of every number, whether it's positive or negative, produces a positive quantity. So there *can't* be any real number that could be the square root of a minus quantity.'

The dialogue continues inconclusively. Eventually, Beineberg becomes exasperated:

Beineberg – 'Why shouldn't it be impossible to explain? I'm inclined to think it's quite likely that in this case the inventors of mathematics have tripped over their own feet. Why, after all, shouldn't something that lies beyond the limits of our intellect have played a little joke on the intellect? But I'm not going to rack my brains about it: these things never get anyone anywhere.'

So Törless goes to see his professor for help. After presenting his doubts, the professor replies;

'I am delighted, my dear Törless, yes, I am indeed delighted, [. . .]
Your qualms are indications of a seriousness and a readiness to think for yourself, of a . . . h'm . . . but it is not at all easy to give you the explanation you want . . . Now, you must not misunderstand what I am going to say.
It is like this, you see – you have been speaking of the intervention of transcendent, h'm, yes – of what are called *transcendent* factors, [. . .]
You know, I am quite prepared to admit that, for instance, these imaginary numbers, these quantities that have no real existence whatsoever, ha-ha, are no easy nut for a young student to crack. You must accept the fact that such mathematical concepts are nothing more or less than concepts inherent in the nature of purely mathematical thought. . . .
My dear young friend, you must simply take it on trust. Some day, when you know ten times as much mathematics as you do today, you will understand – but for the present: believe!
There is nothing else for it, my dear Törless. Mathematics is a whole world in itself and one has to have lived in it for quite a while in order to feel all that essentially pertains to it. . . . '

The example of Törless can be replicated. Although placed in the beginning of the century, when profound questionnings were commonplace, the depth of the questionning is not untypical of that of today.

The teachers

The rapid expansion of secondary school systems all over the world has brought into the teaching profession many from other walks of life. Consequently, the recruitment of teachers for an expanding secondary system result in a high percentage of unqualified teachers joining the system, and the programmes designed to train them are poorly designed and ill-equipped. The solutions to these problems require the training facilities to be expanded; they require a systematic in-service training facility, and the incorporation of methods and techniques of selection and training that have proved effective, as well as providing the additional resources that are necessary to assist teachers to improve the quality of their teaching. All this must add to the growing cost of expanding opportunity for secondary education.

In many countries, an attempt is being made to adapt the content of secondary education to the expected needs of those leaving school to take a job. Some are diversifying the curriculum so as to introduce practical or occupational subjects into an otherwise completely academic programme. Naturally, the profile of teachers needed for a diversified curriculum changes profoundly. In mathematics, in particular, the situation is very critical. For historical reasons, 'practical mathematics' or 'useful mathematics' has been largely absent from the academic curricula of secondary schools, and practising teachers are completely unequipped to meet the needs of mathematics in a diversified curriculum. We see, as a consequence, the unpracticability of making major changes in secondary education to meet the rapid changes of society. Casualties ensue, and, as an example, the major educational reform in Brazil, which called for a form of secondary education universally aiming at a professional qualification, was revoked after ten unsuccessful years of attempting to implement it. The current system is a return to the academic model, with optional professional training. This demonstrated the fact that the large majority of students choose the academic track in the hope of continuing towards a university, even if the job market strongly suggests that employment prospects are best for those who take professional courses in secondary schools.

There are other factors quite unrelated to mathematics which impinge upon secondary schooling. For example, in planning for the educational needs of the secondary level, we cannot ignore the drop in the birth rate in developed countries, and to some extent in developing countries. As Törsten Husén points out, 'the pill took educational planners by surprise' (Husén, 1982, p. 48). The combination of demographic change with rapid changes in mathematics itself makes it very difficult to predict the consequences for the secondary level of education. In all probability, retraining will become a major issue. Furthermore, we have to consider the fact that the average age for school teachers is decreasing. This means that fewer new teachers will be taken into school systems. Already, in the United States, the average age has fallen to 41, so that teachers now in action will remain so for about 20 years more, becoming less and less in touch with mathematics as it develops during this time. In this context, it is relevant to remark the dearth of attempts among mathematicians to predict future developments of their subjects. Exceptions are to be found in the recent book by Philip J. Davis and Reuben Hersh (1981), and in a brief paper by Mircea Malitza (1982). These hint at a new kind of mathematics, strongly relying upon computers and having a broad interdisciplinarian flavour. Both characteristics are almost universally strange to contemporary secondary school teachers of mathematics. Even if it is conceded that changes at the primary level will be less profound – which is doubtful – there is no doubt that changes at the secondary level will be dramatic.

The updating of teachers of mathematics for the secondary level requires three important components which traditional training has practically disregarded: modelling, interdisciplinarity and social studies of mathematics. By 'social studies of mathematics' we mean its social history, philosophy and a critical appreciation of the role of mathematics in development. We will elaborate this theme below.

Secondary education

While conceding the importance and priority of primary education, we have to recognise the fact that secondary education is the most critical level in most school systems. This is psychological and due to social reasons. If one asks what the expectations are of children who are about to finish compulsory primary education, the invariable response is a wish to continue in education. Then we have to agree with the two major objectives of post-primary education, as pointed out by Manzoor Ahmed (Ahmed, 1983, p. 38): '(a) to facilitate the process of adjustment to adult economic and social roles, and (b) to offer opportunities for continuing education at least to those who have the motivation and the drive.' Motivation and drive result from social pressures. A family who succeeds in sending one of its representatives through the school system up to secondary school level will not wish to stop there. All the efforts, both of the individual and of the supporting family, will be directed towards sending him or her to higher education and up to a university. Any reforms of secondary education of an élitist character will conflict with the expectations of parents and students alike. They will bring in their wake an immediate request for better preparation to pass competitive examinations for entry to the universities, rather than for help to find better employment or for a better curriculum designed to fit the young individual to his or her social and economic role in society. For mathematics, in particular, this presents a serious obstacle to bettering secondary education. It militates against providing a more flexible programme and introducing cost-reducing measures.

Mathematics is invariably identified as the core discipline for scientific and technological careers. And these are more and more associated with a better social position and with more stable and secure employment. Mathematics, too, plays a central role in new patterns of employment. It is widely accepted that job opportunities will grow in the information-related industries. These are also invariably associated with mathematics. The growing pressure to make use of computers at all levels in industry and commerce also calls for mathematical know-how. The very name 'computers' in the English-speaking world calls to mind the word 'computation', and hence 'mathematics'. This, however, calls for comment, mainly on misconceptions. While computers can indeed do mathematics, and can be a helpful tool in school mathematics, they have the power to go beyond the scope of mathematics just as mathematics itself goes much further than what computers aim to perform. Calculators, in a way, have a much closer identification with mathematics, because the potential of calculators is clearly associated with quantification. Just as reading and comprehension have been a basic issue in primary mathematics, so computers and programming will become a basic issue in secondary mathematics.

The curriculum of the primary schools of the future seems likely to embrace three main elements: 'instrumentation' subjects, content and socialization. 'Instrumentation' subjects are those skills which are taught merely as 'tools' or 'languages'. They are calculating (with calculators!), modelling, reading, writing, information-retrieving and simulating. This 'electronic era elementary quadrividum' will provide the necessary tools for the individual to pursue further studies. Ability to quantify is a necessary tool for better modelling, and what used to be done by elementary geometry can, with modelling, give classical geometry its real scope and effectiveness. Reading and writing cannot be dissociated from retrieving computer-stored information. And this, together with simulation techniques, already present in video-games, means introducing the child to the modern electronic environment. As to the traditional 'information giving' characteristic of schools today, this will become the role of the media. The teacher will cease to be the 'transmitter of knowledge'. Instead, more up-dated knowledge will be provided directly by sources of the traditional kind (e.g. printed) and by non-traditional means (electronic-based). The latter will include television as an important source of information.

The third component, socialization, will depend on modifying the school environment: More team work, more research projects for students, more questionning of real situations. This model of the primary curriculum in the electronic era has been hinted at elsewhere. (D'Ambrosio, 1979b, p. 27, figure 1). Figure 1 summarizes this concept of curriculum.

All the indications we have point to the development of primary education of the kind described proceeding at an accelerating pace. If so, new entrants to secondary schools will be affected by profound changes in their elementary schooling. So, students now being prepared to teach in secondary schools may well face differently-prepared adolescents at the beginning of their professional life or soon afterwards.

But waves of change do generate repercussions. Parents react, and social pressures build up. Conflict ensues – with all the uncertainties this implies. No parent would remain supine to measures intended to innovate mathematics teaching. The 'back to basics' movement in secondary education reflects the concerns of ambitious parents and students alike. 'Basics' are the surest tools for passing further examinations.

To summarize, the further we go up the educational ladder, the stronger are the conservative pressures, particularly among the less privileged groups in all societies. So, while we can confidently predict the growth of computers in primary education, we can assume increasing reaction against innovation in secondary mathematics education. Not surprisingly, innovation takes better root in upper-class schools: this tends to reinforce class privileges in a world of changing economic

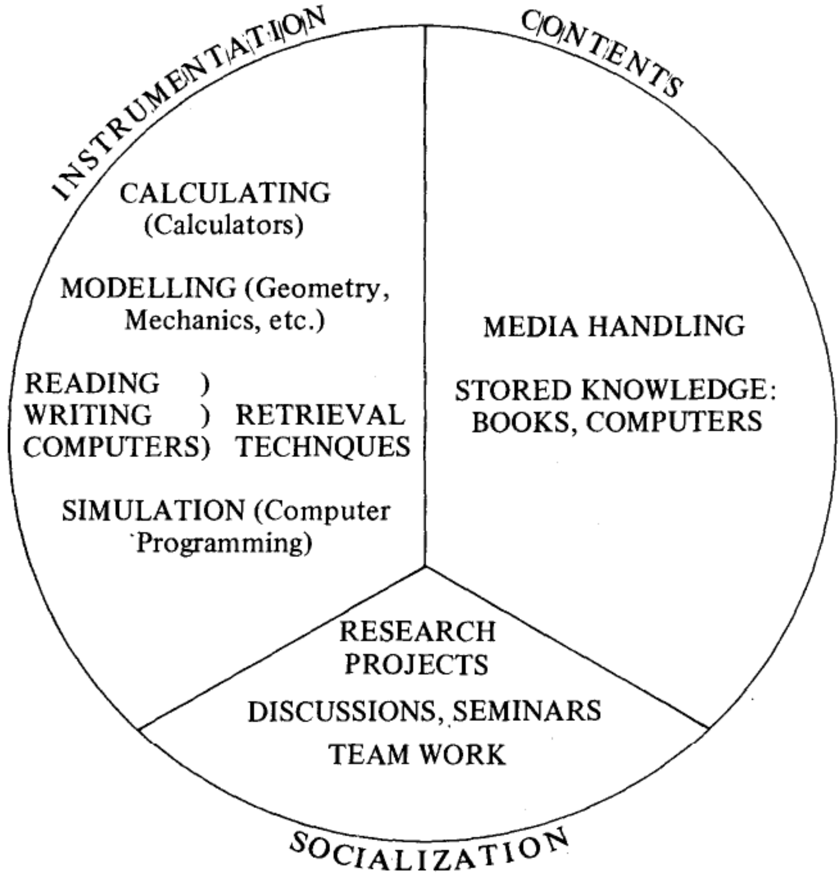

Figure 1. An elementary school curriculum for the electronic era.

and managerial structures (both political and social). Those who are most likely to be leaders in the future are the children of those already in command. The conservative attitudes of a growing secondary education system is the main cause for this. Ironically, parents handicap their children by expecting them to be like themselves. This is particularly serious for secondary school mathematics, where the most profound changes are foreseeable. To meet the challenges of mathematics education in the closing decades of this century requires a breaking with conservative attitudes towards the subject. The key issues seem to be the elimination of the polarization there now is between such dichotomous categories as pure and applied mathematics, skills and comprehension, concrete or abstract approaches, intuitive and formalized approaches, and many others. Instead, innovation calls for a deeper dialectical approach to mathematics, in close relation to the scientific disciplines.

There is no room other than to trigger intellectual action from the environment. This leads us to discuss the role of the environment in mathematics education, specifically at the secondary level.

The environment

We will understand 'environment' in its broadest connotation. The Random House dictionary defines 'environment' as 'the aggregate of surrounding things, conditions, or influences, esp. as affecting the existence or development of someone or something'. Similarly, the Merrian-Webster 3rd International Dictionnary gives 'environment – the surrounding conditions, influences or forces that influence or modify, as (a) the whole complex of climatic, edaphic and biotic factors that act upon an organism or an ecological community and ultimately determine its form and survival; (b) the aggregate of social and cultural conditions (as customs, laws, language, religion, and economic and political organization) that influence the life of an individual or community'. These cover the concept we have in mind for 'environment'. It is very close to what we have called 'reality' in earlier papers (D'Ambrosio, 1981).

We are thus led to consider cultural issues as important components in our discussion. We will examine how these issues, together with social determinants (such as under-development, poverty, colonial practices, social inequities, job insecurity, lack of freedom, etc.) and physical conditions (such as climate, vegetation, degree of urbanization, etc.) exert an influence. And we shall try to identify some differences and similarities in the situation for the poor and the rich parts of the world, be it within a single country or between one country and another.

We must assume, if for no other reason than convenience, that we are talking about the same mathematics all over the world, with the same notation, the same definition and the same theories, with some exceptions at the very elementary level, where we recognize the existence of mathematical practices which differ essentially from one cultural group to another in so far as the local language codifies popular practice and the daily needs and uses of numeracy. In this connection, it should be noticed that while illiteracy is detected quite frequently in the under-developed world, 'innumeracy' is very rare – almost as rare as an inability to communicate.

Illiteracy means an inability to read or write. It refers to a language which may be strange or remote to the practitioner. The language may be foreign, in the sense that it is unknown to the illiterate even in its spoken form. This would be true when we talk about the language of the former colonizer. But it is also true of the official and only language of a country. It is also true of countries where language takes different norms according to social classes. The same language can have such subtleties of use, that the concept of literacy must be interpreted in a broader sense. Moreover, we have to associate the concept of literacy with the availability of reading material.

With respect to numeracy, the above considerations take a different form. In discussing literacy, we have to recognise the fact that the two languages – the mother-tongue or home language, and the 'learned' language – coexist, allowing social groups to communicate among themselves. With numeracy the situation is quite different. The 'learned' numeracy eliminates what might be called 'spontaneous' numeracy. When an individual, who can manage perfectly well the numbers, operations, geometric forms and notions he uses, faces a completely new and formal approach to these same facts and skills, he or she experiences a psychological blockage, which separates the different modes of numerical and geometrical thought. Clearly, in these matters social interest is less frequent, and, in many cases, involves communication with individuals from different strata of social and professional life. Consequently, there is a loss of traditional ways of doing arithmetic and geometry which, however, is retained by those people who never go to school. On going to school, many children lose the capabilities they had, but are not able to replace them with the 'learned' skills. So, we conclude that the early stages of mathematics education can be for some children a very efficient way of instilling a sense of failure and of dependency. Social communication about economics, prices, money control through borrowing and salary negotiations, taxes, constructions and village life, as well as urban planning and administration (which all depend on elementary arithmetic and geometry) becomes much more difficult. Decision making depends on the few who have passed through the school system.

Returning to the possible disruptive social effects that mathematics education may cause, we have to recognize the growth of technology – particularly of machinery, fertilizers, even of processed food, medicine and media. All of these depend, strongly, on mathematical competence, even if the required competence is very elementary, such as reading instructions, turning knobs to the right channel, comparing prices and the contents of fertilizers and food packages. Again, the mathematical competences learnt at home and which are lost in the first years of schooling, are essential for everyday life and labour. The former spontaneous abilities have been downgraded, repressed and forgotten, while the learned ones have not been assimilated, either because of a learning blockage, or as a result of dropping-out early from school or of failing or for one of many other reasons. The individual is clearly 'innummerate'. He or she depends on someone else to cope with the growing presence of mathematics in his or her everyday life. He or she is now more dependent than before going to school.

This is just as true for more advanced topics, in secondary schools, in university and in research. Beginning in the primary school, where a new formalism replaces a spontaneous way of dealing with numerical and geometrical situations, a gap begins to open between culturally-rooted everyday life and practices and school practices and modes of thought. It is possible, and in many cases it happens, that some individuals are able to replace, with some effectiveness, the former, spontaneous way of dealing with the numerical and geometrical facts of his reality by a formally structured body of knowledge and appropriate mathematical practices. But these new and formal practices are the product of a developed body of mathematical knowledge, originated and cultivated elsewhere, in a different reality. So they implicitly carry models and modelling practices which derive from alien experiences and expectations. As a result, the way the individual senses reality, his reality, begins to be affected by this formalized mode of thought. Inevitably, therefore, his planning of action, be it a purely cognitive action, as in the process of learning, or an action with the immediate aim of modifying reality, will be deeply affected in an increasingly alienated way. His power of communicating with his own reality begins to decline, just as the sophistication of his language, the modification of his vocabulary and of his manners make difficult his social communication with culturally akin individuals.

The cycle (reality-individual-action-reality) illustrated in Figure 2 is

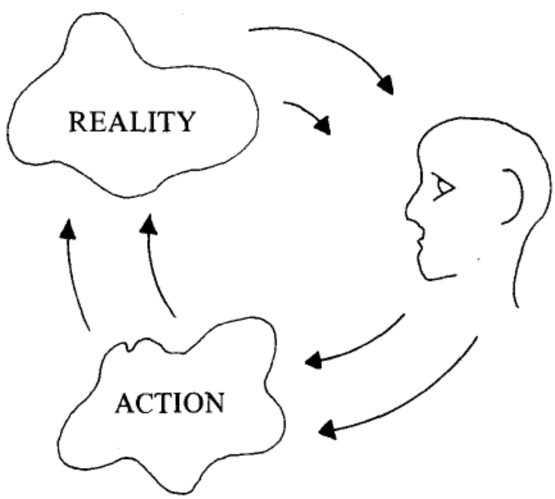

Figure 2. Human Behaviour.

affected by a modification of the individual's internal logic. This is due to new forms of language and codification, as, for example, mathematical codification. Consequently, the individual's more advanced mathematical techniques and his or her research motivation and interest, which grow out of the formalization which underlies the process of becoming mathematically literate (in the sense of acquiring the knowledge and techniques of the established mathematical science) and of the development of a special terminology and structured way of thinking, become increasingly alienated from his or her reality, that is to say, his or her environmental or natural and physical reality, but, perhaps, even more importantly, of his or her social and cultural reality.

The question naturally asked then is: should we, in order to avoid this alienation, abandon school mathematics? Clearly not. No more than we would abandon a foreign language in order to communicate among ourselves. This is because we know that its acquisition does not exclude sharing our thoughts, our feelings, our expectations with our kinsfolk, in our traditional mother-tongue.

My concern is with the compatibility of cultural forms, with reducing to a minimum the possibility of conflict which is inherent in the above analysis. Historical and geographical circumstances have placed many people in the difficult position of learning and living in two distinct cultural realities. The mathematics in schools must be of such a nature that it facilitates knowledge and understanding, while incorporating and making compatible the traditions of the indigenous society. In other words, there must be recognition of and an incorporation of 'ethnic mathematics' into the curriculum. So the scheme given in Figure 3 becomes much more complicated. Indeed, the (S,T) which are the socio-cultural pre-conditions, and which are the basis of modern curricular development, are, in this case, of much greater complexity. The already complex correspondence of Figure 3, where the arrow represents the challenge of modern curriculum development, now has the added complexity of incorporating to it the 'ethnic' mathematical component. This component will call for quite difficult anthropological research in mathematics, a field as yet poorly cultivated. Together with the social history of mathematics (which aims at understanding the interplay of socio-cultural, economic and political factors in the development of mathematics), mathematical anthropology, if we may coin a term, is an essential theme in third world countries. In this context, they are not to be seen as academic exercises, but as uncovering ground upon which curriculum will be developed. Of course, curriculum development in third world countries needs also a more global, certainly a holistic approach. It should not only consider methods, objectives and content in solidarity, but also incorporate the results of anthropological findings into the three-dimensional space which we have characterized as the curriculum.

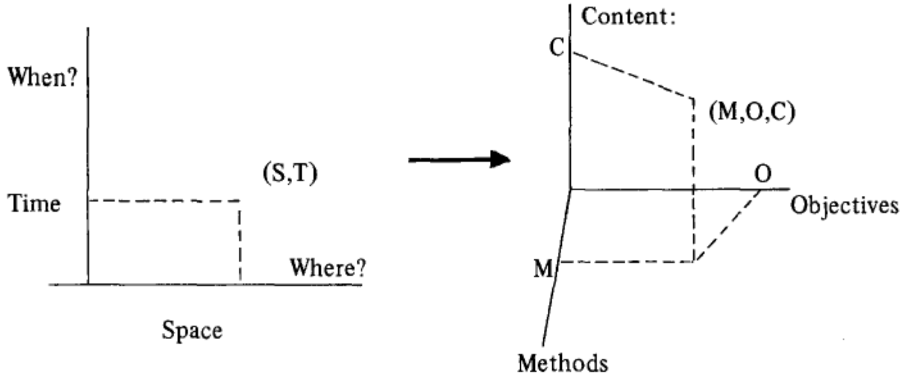

Figure 3. Socio-cultural implications for the curriculum.

These remarks on research priorities in mathematical education for third world countries have obviously a counterpart in the development of mathematics as a science. Clearly, the distinction between pure and applied mathematics has to be interpreted in a different way. What has been labelled 'pure mathematics', and continues to be considered as such, is the natural result of the evolution of the discipline within a social, economic and cultural atmosphere which cannot be disassociated from the particular expectations of a certain historical moment. It cannot overlook the fact that L. Kroenecker ('God created the integers – the rest is the work of men'), K. Marx, and C. Darwin were contemporaries. Pure mathematics, as opposed to 'mathematics', came into focus at about that time, with obvious political and philosophical undertones.

Environmental factors in mathematics education

Let us now turn to some practical considerations of how the environment can be taken into account when practising mathematics in secondary education.

It seems an unquestionable trend all over the world to try to relate mathematics to real situations. Many educators even use, to my view in a distorted way, the concepts of 'back to basics' and 'problem solving' with this end in mind. They imply, in fact, that their objectives are 'useful mathematics' and 'mathematics for everyday life'. This is not borne out in practice from what we can infer about the way 'back to basics' and 'problem solving' have gone.

Consider the concept of 'applications of mathematics'. One thinks of 'reality' as the starting point. Reality, in its full context, has a variety of features: socio-cultural, physical, phenomenal, and even psycho-emotional, as does the environment in the way we have used it above. This means that an individual is immersed in a reality of his own. His feelings, his interests and his perception of reality are very personal. The complexity of situations that are real, i.e. belong to a person's reality, cannot be denied. The depiction of reality is a very personal thing and the parameters one distinguishes in this depiction are privileged, according to values, intentions and even contingencies. This does not in any way deny the material nature of the real world. The perception of its reality is the key issue in our approach to education. At the same time, it is undeniable that man does not submit to this reality. It is proper to our species to act upon this reality. The most striking characteristic of our species is the non-submissiveness associated with action, specifically *intelligent* action. This action, even when it is purely cognitive, modifies reality. The material nature of the real world changes as the result of continued action. The perception of the real world changes as the result of cognitive action. And action is generated by the perception of reality and non-conformity with it.

Now we come to the crux! How directed can this action be? How much direction is acceptable in guiding the perception of reality, which will generate the actions? School systems provide the mechanisms for direction (D'Ambrosio, 1981). Mathematics education is a particularly strong directing mechanism. Mathematics, as with every codified discipline, has its rules, signs, basics, and its effectiveness in explaining real situations, or in solving problems, derived from the use of accumulated accepted knowledge, and the handling of necessary techniques. This, of course, is preceded by ingenuity, creativity and insight. The first group, namely ingenuity, creativity and insight, must be cultivated like a plant. They need to be sown in suitable soil, and they must be nourished. Here, fertility is determined by motivation, curiosity and initative. We cannot deny that formal school systems fail to develop such qualities. I would even allege that, in fact, they work in the opposite direction. Excessive authority, identified with the language and even the posture of the teacher, reinforced by culturally absurd mechanisms of evaluation, kills the natural components of insight, creativity and ingenuity with which children arrive in school (D'Ambrosio, 1983). We are gathering results which show that mathematics as a school subject has been responsible for much of this state of affairs. Particularly striking evidence has been obtained by Eduardo Luna in the case of Latin America, within the framework of the Second International Association for the Assessment of Educational Achievement (IEA) Study in Mathematics (Luna, 1982). He has shown that the ability to perform the basic operations, one of the components most mistakenly sought for in the 'back-to-basics' movement, is completely unrelated to functioning intelligently in new situations. It is also disturbing to realize that problem-solving does not improve the situation, at any rate in the way it is traditionally taught (more as a drill in solving stereotyped problems, artificially treated in order to cater for the immediate application of the basic operations, and usually set so as to give 'round' results).

In looking into what is done to improve the situation in Latin America, we are led to an overall analysis of the complexity of factors which education faces in the region. Many factors can be identified. Indeed, a project is underway designed to define the socio-cultural variables in mathematics education. This is particularly important as soon as we realize that a form of ethnic mathematics exists, with many of the characteristics of what we would call 'applicable' mathematics, and which is practised by children and adults 'up to entering school'. Then a conflict arises which disables the children's mathematical functions in the real world, as we pointed out in the previous section. A fact to be investigated is that introducing calculators presents fewer, probably none, of the effects mentioned above. In other words, commonplace practices permit an easy transition to the use of calculators, while school practices (operations with paper and pencil) virtually kill any previously-known mathematical skills. This phenomenon is now under analysis from the viewpoint of cultural anthropology.

In most Latin American countries, curricula are much the same all over the country. The major cities serve as the models. In these, largely urban populations have expectations of the schools. They either regard them as a means towards social access, and hence a route to college education, or as a means to fulfil basic legal requirements for lower middle class employment in the public services, commerce, banks, etc. For these, more than primary education is rarely required, and post-primary standards are minimal. On the other hand, those who aim at college-level degrees look for an education that enables them to pass highly competitive entrance examinations to the universities. The curriculum is strongly influenced by what is required in these examinations, and secondary education is dominated by these requirements, which are classical, and based on training and drilling for multiple-choice testing.

A few examples of attempts to introduce more lively and creative programmes can be found. These are isolated cases, and cover interesting applications of the most varied nature. In particular, there are projects which bring the concept of modelling into the very early years of schooling, dealing with real life situations. Problems dealt with at the primary level are, for example, the construction of kites and the construction to scale of houses, cars, etc. The novelty is that 'theory' goes together with 'doing'. Measures are taken, reduction of them takes place, material is bought and an object is the final product. This activity is more in the line of a 'project', coupled with 'theoretical' reflections at every step of the process. But such projects are restricted to a few research groups and special training centres. They can hardly be seen and known about in most of the countries. At the secondary level, the challenge seems to be to make what is required in the entrance examination to the universities compatible with more creative and interesting mathematics. One source of examples can be problems taken from newspapers – analysing the financial news, for example.

There is a particular need to concentrate effort at the graduate level, where trainers of teacher are prepared. Many seminars and workshops should be held on applied mathematics for college teachers who have responsibility for the training of prospective primary and secondary school teachers. This is urged since the highly traditional and theoretical curricula in the teacher-training courses are unlikely to produce teachers able to stimulate work other than routine and trivial applications in their classrooms. The use of calculators should be encouraged, particularly in the calculus and numerical analysis courses of universities. Statistics should be always present in teacher training-courses.

We notice that the less a country is developed, the more formal and theoretical is its mathematics teaching. As mentioned elsewhere, in spite of protests from conservative teachers and parents, the sale of calculators has become a thriving business, even in the developing countries, and children manage them with ease. Calculator games, play-rooms and other electronic ambiances favourable to mathematics-oriented machinery are frequent. This engenders a growing facility to deal with mathematics. In fact, it is no longer a secret what the core of the basics should be. Hence, teachers are afraid of being substituted by calculators!

The most effective example of mathematics being related to the environment, both the socio-cultural and the material environment, is to be found in the so-called 'ethno-mathematics', mathematics ethnically related to particular communities. Although it is, as yet, unrecognized, 'ethno-mathematics' is being practised by uneducated people, sometimes even by illiterates. Some research effort is being made to identify 'ethno-mathematical' practices and to incorporate them into the curriculum. But there is much work to be done before 'ethno-mathematics' becomes recognized as valid mathematics and becomes accepted as a valid component of pedagogy.

A brief review of projects and research

The basic reference source for cultural factors affecting mathematics education continues to be the bibliography collected by Brian Wilson, (1981). And, in the context of mathematics and language, a basic issue of environmental factors, the survey paper (J. L. Austin and A. G. Howson,

1979) on Language and Mathematics should also be mentioned. Some seminars have been conducted in Africa on the relationship of language and mathematics; all of them have important references to environmental issues. The landmarks have been the meetings in Nairobi (Symposium on Interactions between Linguistics and Mathematical Education, 1974) and in Niamey (Séminaire mathématiques, langues africaines et langue française, 1977). Particularly important was the colloquium on Mathematics and Environment which took place in Abidjan (Colloque inter-IREM : mathématiques et milieu en Afrique, 1978). In 1976, an important meeting was held in the Caribbean, with particular emphasis on science education and teacher training (Seminar on Science Education Projects in Caribbean Countries, 1976).

A key issue of the role of the environment in mathematics education is closely related to the concept of integration in science education. This has been discussed in two papers (D'Ambrosio, 1980 *a*; D'Ambrosio, 1982).

Several projects designed to explore the concept of 'ethno-mathematics' have begun to appear. This, in our view, is the link between traditional, academic mathematics and environmental mathematics – understanding 'environment' in its broadest sense. Many of these projects are published in the *UMAP Journal* (Undergraduate Mathematics Applications Project, Cambridge, Mass.) and the *Bulletin* of the Institute of Mathematics and its Applications (Southend-on-Sea, Essex). We have selected for mention these two journals, though representative of the developed world, because they are fully devoted to environmental issues in mathematics education. We should mention also books by H. Burkhardt (1981) and B. Spotorno and V. Villani (1976) as very good sources of examples.

Research involving environmental aspects of ethnography are more and more frequent. And research projects in mathematics education are becoming more and more common. Learning theories reflecting environmental issues are an attractive field for research in developing countries. I refer, in particular, to the important and profound work on cognition undertaken by the Centro Interdisciplinario de Investigaciones en Psicologia Matática y Experimental [Interdisciplinary Centre for Research in Experimental and Mathematical Psychology], in Buenos Aires (CIIPME – Cangallo 2158, Buenos Aires, Argentina).

Several curriculum development projects in mathematics related to environmental issues are provided in Latin American countries under the aegis of the Centro Interdisciplinar para a Melhoria do Ensino de Ciências (Interdisciplinary Centre for the Improvement of Science Education) in Campinas, Brazil: (CIMEC: Caixa Postal 6063, Campinas, SP, Brazil).

In particular, 'ethno-mathematics' is a field of growing importance. Not much is yet published. An important resource for Latin America is

Chamorro (1983). There, one finds reports on several projects in the education of American Indians. Although not yet reported, there is the important Proyecto Experimental de Educación Bilingue-Puno (Experimental Project on Bilingual Education – Puno) conducted in Peru by the Ministry of Education through INIDE: Apartado 1156, Lima 100, Peru. While it deals mainly with primary education, the project has a teacher-training component which, in most developing countries, represents a major component of secondary school systems. Termination of secondary school systems is almost totally identified with teacher-training programmes.

Particularly important is the work being done at the Department of Mathematics in the Faculty of Education of the University of Mozambique. This is reported in the periodical TLANU (Caixa postal 2923, Maputo, Mozambique).

In an International Seminar sponsored by Unesco in 1980 in Huaráz, Peru, some examples of the training of teachers of mathematics to deal with environmental issues were discussed (D'Ambrosio, 1980).

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Ed Jacobsen

Reducing differences of mathematical expectations between boys and girls

There were 186 competitors from 32 countries, including 7 girls, proclaimed the report of the XXIV International Mathematics Olympiad, Paris, 1983.

The same dearth of women with high qualifications in mathematics is reflected in most countries by a scarcity of women in those jobs which require high competence in mathematics.

Dimensions of the problem

How has this situation occurred? Since the achievement of both boys and girls is combined in the data of most countries, only some examples will be mentioned. One well-known statistic is that 51 per cent of males, compared to 8 per cent of females, in the incoming first year class at the University of Berkeley (California) had completed four years of secondary school mathematics (Sells, 1973). Thus, those courses of study with a mathematics pre-requisite at that university were closed to 92 per cent of the incoming women students, unless they first took remedial mathematics classes. The National Assessment of Educational Progress (NAEP, 1980) reported that, at age 14, schoolgirls in the United States compared equally in mathematics achievement with boys, but, by age 17, the majority of girls no longer study mathematics, and those who do continue average lower achievement scores than boys. In the United Kingdom, the Assessment of Performance Unit (APU 1980), which monitors attitudes and performance in mathematics of 11- and 15-year-olds, showed the same differences, both in attitudes and in performance as noted by NAEP above. But it found them already beginning at age 11.

In research carried out as part of the Second International Mathematics Study, Schildkamp-Kündiger (1982) documented the same sex-related differences in mathematics achievement in a wide range of countries of differing economic levels. In general, in most countries, girls drop mathematics as early as possible, thereby precluding themselves from taking mathematics at university level, and excluding themselves from many careers. A look at many 'male' professions confirms this.

Let us look at these differences in performance in more detail, then at what may be the causes, and finally at what can be done to improve the situation.

The influential 'Cockcroft Report' on school mathematics in the United Kingdom contains an annex (Shuard, 1982) on the differences in mathematical performance between girls and boys. Shuard examined reports from APU, NAEP, Wood, Fennema and other researchers in this field. The differences may be summarized as follows:

Girls were better on computation with whole numbers, on entirely verbal items involving naming geometric shapes and on making a deduction from given verbal non-numerical information;

Boys were significantly better at solving word problems, at applications of number, measurement, rate and ratio, place-value and at spatial visualization.

Thus, girls out-performed boys on the type of test items which are susceptible to drilling or which require verbal ability (except verbal mathematical problems), while boys excelled on those items which required problem-solving behaviour.

Causes and concerns

Researchers generally agree on some important psychological differences: that girls generally show a lack of confidence in mathematics, and believe that to be successful in it is unfeminine; that boys attribute success in maths to their ability and failure in it to bad luck or to lack of trying hard enough, while girls feel just the opposite – that success is due to extra hard work, or to good luck, and failure to lack of ability. Thus, if boys are not doing well, they feel they should make more effort, while girls feel they might as well quit, as they lack ability. These psychological variables both suggest the cause of the problem and a possible approach to its solution.

These same attitudes, which lead to giving differing reasons for success or failure as between boys and girls, are also widely shared by teachers. Indeed, it is argued that the mathematical expectations which teachers hold for boys and girls is manifested in differential treatment on the basis of sex in mathematics classes. The result is that students respond differently in class, so 'justifying' the teachers' attitudes.

What form is this differential treatment? It may be subtle and entirely subconscious, such as was revealed in a study (Gore and Roumagoux, 1983) showing that the length of time a teacher waits for a student to begin responding to a teacher-posed question in mathematics differs between boys and girls – that teachers give significantly more 'wait-time' to boys than to girls. Some studies of classroom interaction have shown that girls' mathematical ideas are not as carefully listened to; that a boy's partial answer will be prompted by the teacher to be developed further, while a girl's will not be. Another study (Becker, 1981) showed a differential treatment as between boys and girls on the cognitive level of questions posed by the teacher, in the teacher's praise and criticism, and in the individual help given. All of this strengthens the idea that mathematics is a male domain. And so it has become.

It is comparatively recently that attention has been given to this problem. Typical is a survey by the Rheinland-Palatinate Ministry of Education in the Federal Republic of Germany, made as a result of the deteriorating employment prospects for young women, with automation reducing the number of traditional female occupations, such as clerical and light assembly work. The survey (Dungsworth, 1981) confirms that sex-related preferences for mathematics and science subjects are already established in the lower school, and that, by the end of senior secondary school, boys far outnumber girls in those classes (87 per cent in the terminal year of the physics course were boys).

The survey has prompted an allocation of funds for research to discover the reasons why girls restrict their range of subjects and to determine how far teachers, parents and changes in the curriculum can encourage them to broaden their choice. Here, the interest is not so much in the differences in performance between boys and girls but in the recognition that girls tend to be less well-prepared to enter occupations and careers that require knowledge of mathematics and science, and that mathematics is the 'block' preventing females from entering studies required for many professions. Indeed, the desire for new roles for women in society can only be realized if a greater number of girls study mathematics to higher levels.

At the same time, many societies cling to the idea that mathematics is a male domain. The attitudes of teachers, which were discussed above, are just a reflection of society-at-large. Child-rearing practices condition, at an early age, a person's career expectations. Peer group pressure maintains these expectations, so there is little objection when the student is told that she cannot take advanced mathematics or physics because it clashes with home-economics, health or biology on the school timetable. In the USSR, the entrance examinations of the special mathematics secondary schools, the Pioneer Boarding Schools started by Kolmogorov, 'the competition is open of course, to both sexes, who have had exactly the same primary education. But, as a matter of brutal fact, the overwhelming proportion of successful candidates are boys' (Snow, 1969, p. 507).

It has been argued that the difference in mathematical attainment between boys and girls is due to basic biological differences in the sexes. At the XXV International Mathematical Olympiad in Prague 1984, in discussing reasons for so few female competitors, the author was firmly told by the team leaders from two countries that the percentage of women competitors can never exceed 8 per cent, as this is due to biological reasons which we simply cannot alter. What is the evidence for this? And does it really matter if the causes are social or biological?

Research has concentrated on one cognitive area where boys consistently out-perform girls – that of spatial visualization (the ability to mentally manipulate objects in space). A complete review of this research would alone fill this book. Suffice it to say that, however the tests are designed, the average scores for boys are higher than those for girls. But, might the same social factors that cause differences in mathematics achievement cause differences in spatial visualization? Fennema and Sherman (1978) have taken a different approach. Their examination shows that spatial visualization plays an important part in learning geometry. But then they specifically investigated the relationship between mathematics achievement and spatial visualization skills and concluded that the data do not explain the differences found in mathematics achievement. And, asking if the very small difference in spatial visualization between boys and girls could possibly account for the large sex-related differences in achievement, they concluded that it could not be. The argument that biological differences affect mathematical learning disappears when programmes designed to remove the social factors actually improve the girl's attainment in mathematics, or that when both males and females study the same amount of mathematics their differences in accomplishment are lessened.

Except for the persistent, but slight, differences in spacial visualization, other differences in performance tend to disappear with equal treatment of boys and girls. This makes untenable a 'biological difference' argument. But, to return to the question of whether it really matters if there exists a biological cause, the answer is certainly yes, for, if a girl believes in an unalterable biological cause, she will easily give up when confronted with difficulties. The research (Høyrup, 1978; Kaminski, 1982) has been invaluable, as we now know the reason why fewer females choose to study mathematics, and why their achievement in mathematics is generally lower than that of males. Let us then examine some intervention programmes which can be used to reduce the social factors that prevent girls from participating equally with boys in mathematics classes.

Single-sex mathematics classrooms

In most countries which reported sex-differences in mathematical achievement it was noted that these differences were greatly reduced in the all-girls schools. This has been reported by Schildkamp-Kündiger (1982) in a survey that included most countries in Western Europe and the United States, but also by Luna (1981) in the Dominican Republic, and by Jacobsen (1976) in Botswana, Lesotho and Swaziland.

In this connection, the Cockcroft Report warns that most single-sex schools are selective. This alone could account for the differences in achievement. But Jacobsen found the differences persisted after removing the factor of intelligence. The conjecture is that girls achieve more in all-girl classes because they will be free from the negative peer-pressure of boys, reminding them that it is unfeminine to be good in mathematics, and that differential-treatment by teachers cannot operate in single-sex classrooms. Thus, a number of single-sex mathematics classes have been established in mixed schools. In one mixed school in the United Kingdom, Smith (1980) studied the results of 18 months of teaching one classroom of girls separately from boys, and another classroom mixed, the same teacher trying to give each an identical treatment. He found that the mathematics performance of the girls in the mixed classroom declined significantly in comparison, both with the boys in this classroom, and also in comparison with the all-girl classroom. But the girls in the all-girl classroom achieved equally with the boys in the mixed classroom. These classes of incoming secondary school students had been matched in ability at the start, but the girls in the mixed class conformed to the usual pattern of the school by falling behind the boys, while the all-girl classes remained equal.

However, after trying single-sex mathematics classrooms in mixed schools for a few years, other schools in the United Kingdom have given them up and returned to mixed-classrooms. Wilce (1984), formerly an advocate of single-sex teaching groups, now claims that the improved performance of girls is due to other factors, and that the single-sex groups create more problems than they solve. These include negative staff reactions, and increased discipline problems in all-boy groups in mixed schools. Fennema (1979) had cautioned about single-sex classrooms, noting the results of past segregation of women and minorities.

The long-term benefits of establishing single-sex mathematics classrooms in mixed schools are not yet proven, and more careful research and experimentation are needed. Meanwhile, it remains a relatively simple solution to a complex problem, but many would not recommend it. In the successful single-sex classrooms mentioned above, Smith (1980, p. 36) stated that the staff had reservations about the programme since classroom discipline was a problem: 'There is obviously far more to it than that [separating the sexes]. Parental attitudes, teacher attitudes – they all favour boys. Single-sex setting probably can be justified as a short-term measure, but you've got to change attitudes way beyond that'. Other special groups have formed to change these attitudes.

Let us now look at some other efforts for improving the situation. They centre around the following strategies: increasing girls' perceived need for and usefulness of mathematics in various careers; reducing girls' mathematics anxiety and increasing confidence; and changing attitudes of others who influence girls (parents, teachers, counsellors, boys, textbook writers and editors).

Career awareness and selecting mathematics

Girls are probably realistic in believing that they will be discriminated against if they are successful in mathematics, and they find it difficult to choose a career that demands behaviour which people generally find attractive only in men.

In the United States, groups such as Women and Mathematics (WAM) produced and distributed to schools booklets giving the names of women scientists who are willing to talk to school groups about their lives and work. Talks by women mathematicians and scientists give the girls the opportunity of learning how a woman can pursue a scientific career and still have a normal family life. The girls learn of the compromises of time and energy involved in balancing career and family, and that a woman scientist can be feminine.

The British Girls into Science and Technology (GIST) group also has a scheme involving talks in schools by women scientists. Smail, Whyte and Kelly (1981) explain how the group advertised for women scientists who might be interested in the programme. Part of their training consisted of reading essays by eleven-year-olds written about an imaginary interview with a famous woman scientist. This acquainted the group with the language level of eleven-year-olds as well as with their conceptions of women scientists. Microteaching sessions allowed each scientist to judge the level of her talk and to determine which areas were most likely to appeal to eleven-year-olds. For example:

> When children are working on forces we have a lecturer in anatomy who can talk about the force in the human arm. When they are learning about acids and alkalis there is a food technologist who brings cakes made from cake mixes with incorrectly balanced baking powder to show the use of pH measurements in the food industry. Learning to use a thermometer is linked to a gas engineer who talks about the different ways of measuring temperature she uses in her work, including a demonstration of heat-sensitive crayons (Smail, Whyte and Kelly, 1981, p. 253).

The Open University and GAMMA (Girls And MatheMatics Association) organized one-day conferences entitled 'Be a Sum Body', advertising them by writing to the special advisory teachers of mathematics. Each girl aged 13 to 16 selected one of fourteen possible workshops, the intention of which was to introduce the girls to aspects of mathematics which might be unusual, unexpected or unfamiliar. Those most popular involved computers, but some others were 'Musical skills and dividing by two', 'maths puzzles', 'maths in nursing'. The concluding plenary session included some women who explained how the use of mathematics in their job advanced their work potential (GAMMA, 1983).

Even after girls are convinced that mathematics is useful and important for their future career, many are convinced they cannot learn it. 'Maths anxiety' and 'mathophobia' are now well-known words to describe the anxiety that many have about learning mathematics. Special workshops have been organized to reduce this anxiety. According to Tobias and Weissbrod (1980, p. 67) the goal is 'to change the classroom atmosphere from one of tension and competition and a resulting unwillingness to ask "dumb" questions to one of trust'. These often begin by making the students aware of the extent to which their anxiety inhibits their learning. By keeping a journal while doing mathematics, they can soon see that their success is usually attributed to luck, but not to ability. Some workshops are devoted only to reducing anxiety, while others combine it with learning mathematics. The purpose is to show that it is the anxiety, not the mathematics, that is the inhibiting factor of learning.

Buxton (1980), in what is probably the best single source on the subject, describes his research and gives suggestions to individuals and teachers for combating mathematics anxiety.

Changing teachers' attitudes

A group in Sweden (Lindholm, 1981) has taken as one of its main objectives the changing of teachers' attitudes toward accepting girls' rightful place in mathematics and science. Thus, teachers must accord to girls' experiences and suggestions the same status as they give to those of boys'. After noting that when thinking about the technological future, boys place very little emphasis on human beings while girls more often describe the everyday social and human situation, Lindholm points out that it is this important contribution that girls can make to help shape a better society.

Since most teachers are unaware of their influence on pupils' attitudes toward mathematics and girls, the MENT-project (Girls in Science and Technology) in the Netherlands (Raat, 1981) demonstrates to teachers the negative position of girls in physics lessons by using videotapes showing how teachers interact one way with boys and a different way with girls. The tapes also illustrate the different ways boys and girls co-operate in lessons.

Researchers from many countries have recommended that girls need a more active, experimental teaching approach to overcome their passive learning. To give an example, the School Mathematics Project requires pupils learning transformation geometry to use paper, drawing pins and mirrors at every stage of its development so that they will always think of geometry as including motion.

To sensitize future teachers to problem areas such as sex-role stereotyping in mathematics teaching, Bishop (1982) utilizes two methods; student group research projects, and simulation and role-play activities. He describes the group projects as consisting of the following: 'individual students could test children on a written-attitude instrument, interview individual children using photographs as stimuli, interview teachers about differences between the boys and girls they teach, observe and record classroom interactions between pupils or between pupils and the teacher, observe non-verbal behavioural patterns in class, and experiment with the use of reverse sex-role mathematics problems. After a time for reflection and analysis, the group discussion of different findings can be a most stimulating experience for all concerned (including of course the tutor)' (Bishop, 1982, p. 129).

The simulations and role-play activities help students to become aware of the feelings, perception, and emotions which are present in the social act of teaching.

Textbooks and girls' achievement

In a number of countries, committees have investigated textbooks for sex-role stereotyping such as showing males in active, professional situations and women in domestic, passive roles, or depicting engineers as always being men, or nurses as only being women.

To counteract the attitude held by girls that success in mathematics for them is a result of good luck rather than ability, textbooks should show that favourable results for both girls and boys are due to individual efforts and actions. Also included in textbooks could be brief biographies and pictures of women mathematicians portrayed as true women and not imitators of men.

The examples in textbooks of war (ballistics of projectiles, missiles) or machines could be replaced with social concepts to coincide with the general values and interests with which girls more readily associate. The Scandinavian countries have been especially active in trying to eliminate all sex discrimination from their textbooks.

When sexism in a mathematics textbook is found, letters can be sent to the publisher pointing out the details, accompanied with any national publishers' guidelines.

Levels of action

Let us conclude by looking at the diversified programme 'EQUALS' at one univeristy, the Lawrence Hall of Science at the University of California. 'EQUALS' provides in-service courses for primary and secondary teachers, counsellors, administrators and parents, aimed at attracting and retaining students (and especially girls) in mathematics. Kreinberg (1982) describes the work as increasing educators' awareness of girls' avoidance of mathematics and the consequences of this and then providing them with materials and strategies for teaching both mathematics and an understanding of its relevance to career options. Those who take part conduct research on the enrolments of girls in mathematics classes to gain an awareness of the problem. Materials designed for immediate use in classrooms are given to participants, along with different teaching strategies. Another part of 'EQUALS' is 'SPACES' (Solving Problems of Access to Careers in Engineering and Science) which consists of classroom activities in mathematics and career education showing how problem-solving relates to scientific activity. 'MATH FOR GIRLS' gives six- to twelve-year-old girls practical work and problem-solving for increasing their interest in mathematics. 'FAMILY MATH' helps the parents to improve their skills in mathematics so that they are better equipped to help their children to learn mathematics and to introduce them to women who are working in scientific and technical occupations. Finally, the 'MATH- SCIENCE NETWORK' associates 1000 scientists, educators and parents who live in the area of the project (San Francisco) with a view to encouraging women to take mathematics courses and to interest them in career opportunities in mathematics and science. This they do by promoting conferences where secondary school girls will meet women working in scientific and technical fields and take part in practical workshops in mathematics and science designed to motivate and challenge them. In this way, this project attempts to attack the problem of girls and mathematics on all fronts.

Girls and mathematics is a problem for which many partial answers have been given. Evaluation has been rare. Stage (1983) suggested at the Fourth International Congress on Mathematical Education (ICME IV), Berkeley, California, 1980, that, since most of the programmes designed to increase women's participation in mathematics were voluntary, because of a scarcity of money, the little money available might better be used for staff or materials rather than for evaluation. Worldwide interest in this topic is so strong that substantial parts of ICME IV and ICME V, 1984, were devoted to discussing it. Actions may be taken at all levels, from those of the parents at home and of individual classroom teachers, to those of the head of the mathematics department and the head of the school, to those responsible for teacher education and the overall running of the schools.

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Barry H. Blakeley

Algorithms, calculators, computers

Introduction

Calculators, microelectronic devices and, in particular, microcomputers are the most recent of the revolutionary aids to education. Teachers in many parts of the world have become used to (and in some cases now ignore) radio, television, teaching machines, overhead projectors, films and filmstrips, all of which have something to offer in the classroom. Calculators and computers are of a different order of importance. In the first place, the microelectronic revolution is vastly different in its rate of making change. Whereas the industrial revolution in the western world took more than a hundred years to make its effects felt, microelectronics is making its impact in the space of only ten years or so; because of its impact on business, shops and the home, the accompanying social change is far reaching. Calculators are now accepted as a feature of life in developed countries, and employment prospects have already been radically affected by the introduction of computers and robots into commerce and industry. The latest boom is in home computers, and these machines, which cost no more than one to two hundred dollars, can do things almost undreamed of ten years ago.

There are other features of this phenomenon which are of prime importance for those who are involved in the training of mathematics teachers. In the first place, there can be no choice about whether or not to pay any attention to it. Developed countries have already shown that children will take to the change quite readily, and teachers who refuse to come to terms with calculators and computers will simply lose credibility with their pupils. Of even greater importance is the fact that the advent of calculators and computers strikes at the heart of mathematics. Even though the bulk of its commercial use is for data handling, the subject matter of mathematics is intimately related to the operation of a computer.

There are three main areas of the curriculum that need to be considered when looking at the impact of calculators and computers on the teaching and learning of mathematics. They are the existing areas of the curriculum, the new items to be brought in and the new methods that are possible in teaching and learning. These have considerable implications for teacher training. There will be many new things for teachers to learn, and, because teachers should be teaching material that did not feature in their own education and using methods which differ from their own learning experiences, the necessary changes will not be easy to achieve.

The existing curriculum

It seems sensible to start by considering the calculator, as it is cheap and is likely to be the first product of the revolution to become widely available in any country. However, it is inevitable that discussion of the calculator will move rapidly into considerations which will apply equally to computers.

The new numeracy

The first point to make is that calculators calculate – and they do it quickly and accurately. It is not enough, however, to accept this fact and simply *allow* pupils to use calculators. Teachers should actively teach pupils *how* to use their calculators. One of the first booklets in the series on the calculator written by the School Mathematics Project in the United Kingdom was called *Discover how to use your electronic calculator.* There are problems to be faced when introducing calculators into schools. What should happen to the 'traditional' skills, those of calculation? What, in an electronic age, do we mean by 'numeracy'? A thought-provoking definition has been provided by Girling (1977, p. 4): 'Basic numeracy is the ability to use a four-function electronic calculator *sensibly*'. At the heart of this definition is, of course, what is meant by the word 'sensibly'. Broadly speaking, 'sensible' use of a calculator means that old skills acquire a new importance, particularly those of estimation, approximation, judgement of the reasonableness of results, accuracy, ability to spot errors, and knowledge of appropriate algorithms.

Many of these are already (nominally) in the curriculum – asking if an answer makes sense, checking an answer by doing the calculation in a different order, making a rough estimate of the expected result. These skills assume a greater importance when calculators are used, particularly as children tend to accept an answer simply because it has been produced by a calculator (or a computer). It is also necessary to strive for 'number sense' – an awareness of arithmetic operations and of number size: about how many words are there on the page of a book, or, about how long is a lorry? Johnson (1978) has been more detailed about the skills required in a calculator curriculum. He suggests that the following 'standard' content is particularly important: an ability with single-digit arithmetic; facility with powers of 10; the understanding of place value;

and these combined with 'number sense' and an ability to use a calculator.

An example will help. When faced, say, with the calculation 620 x 46, the pupil may first use a calculator to produce a result. He or she will then proceed to check the result, thinking as follows:

600 x 40	(using leading digits only)
6 x 100 x 4 x 10	(using powers of 10)
24 x 1000	(using single digit arithmetic and powers of 10)
24000	

It may be noted that a greater number sense, for example a better feel for rounding off, could produce a more sophisticated estimation. However, from this example it is clear that some of the traditional skills are still needed, but the old emphasis on endless practice of long, computational algorithms could and should be removed. Teachers in training (and those already in the classroom) should be involved in objective discussions of how much practice (the traditional skill) is required and how much mental arithmetic (which may take paths quite different from those of the formal computational algorithms) is really necessary for preparing pupils to assume their diverse roles in society.

Even at the apparently simple level of carrying out calculations, the use of the calculator can point to important principles and be the lead-in to important ideas. The concept of 'Input-Process-Output' is an important one in computing. It emphasizes both the importance of the accuracy of the data, and the algorithms used in the 'process'. It is a great temptation to pupils to give an answer to an inappropriate degree of accuracy, simply because the calculator or computer produces so many digits. It is important for them to realize what is appropriate, having regard to possible errors in the data used in the calculation. For example, in a right-angled triangle ABC, where

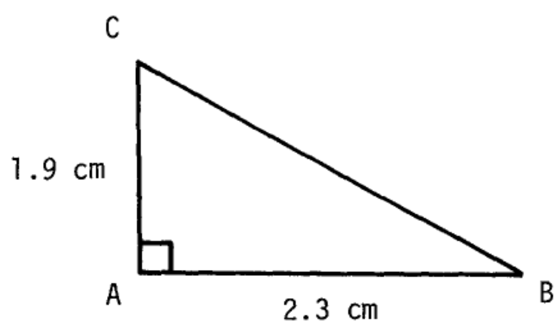


Figure 1

angle A = 90°, AC = 1.9 cm and AB = 2.3 cm, which of these is the most appropriate value for BC?

2.9832867 cm , 2.98 cm , 3.0 cm or 3 cm ?

All four results came from a group of pupils using calculators. The degree of accuracy of the lengths given might suggest that:

$$2.25 \leqslant AB < 2.35 \text{ and } 1.85 \leqslant AC < 1.95$$

Using the extreme values of AB and AC, we find that

$$2.91\ (3) \leqslant BC < 3.05\ (4)$$

And only the last of the four answers given covers this range. Teachers need to be aware of the errors of measurement and the standard ways of dealing with them. The usual procedure is to give the result to the same degree of accuracy as the least reliable piece of data. In the example quoted, this would favour the answer 3.0 cm, which is more precise than the data warrants. Before leaving this example the reader may care to consider the problem if we wish to allow for inaccuracy in the angle quoted.

It is possible sometimes, of course, to improve on the accuracy of the calculator. Consider the following four methods of evaluating the expression $\dfrac{0.00016 \times 473}{2145}$:

$(0.00016 \times 473) \div 2145$ $\qquad$ $(473 \div 2145) \times 0.00016$

$(0.00016 \div 2145) \times 473$ $\qquad$ $\dfrac{4.73 \times 1.6}{2.145} \times 10^{-5}$

Pupils need to be aware that different methods of calculation may produce different numbers of significant figures in the final result. They should realise that calculators and computers calculate to only a limited number of figures (they truncate or 'round off' after producing a fixed number of digits) and because of this it is possible for errors to enter a calculation.

It has already been said that there will no longer be any need for the endless practice of the algorithms for long multiplication and division; the calculator carries out such operations most effectively. We need to go further along the lines indicated by Plunkett (1979), who makes a convincing argument for the acceptance of 'non-standard' algorithms for calculation. When calculating, we need to be flexible in our methods and to allow flexibility when our pupils are doing the same. Probably the most important requirement is that the processes used should be clearly recorded, together with any intermediate results, so that checking or estimation can be carried out, both by the pupil and by the teacher (or the examiner). We shall return to the issue of algorithms in a later section.

The new ordering

Very few calculators will handle fractions as fractions, although some, those which have a key labelled '⌋', will accept an input of

[2] [⌋] [3] [⌋] [4] for 2¾

and will display results in a similar form. The vast majority will accept only an input such as [2] [÷] [3] for ⅔, and will immediately convert this to a decimal. And, of course, all results are displayed as decimals. This raises the question of how much work on fractions should be done, and at what stage in the pupil's development. Some familiarity with fractions is obviously important to ensure an understanding of their everyday use. In addition, the concepts involved are of central importance in work with algebraic fractions. The major implication of these considerations would seem to be that we should do more work on decimals and place value earlier in a pupil's course. Pollak (1977) refers to two partial orderings of the curriculum, one of prerequisites (implied by the discipline of mathematics) and one of importance (implied by the needs of society). A discussion of the interaction of these major concerns must be an important component in the training of teachers for the 'electronic age'.

The new approach

Numbers are not only of concern in arithmetic. They occur in other areas of the curriculum, particularly in the sciences, but also in the humanities. Here, aspects of trigonometry and of probability and statistics spring immediately to mind. The calculator can remove the drudgery and fear of calculation from these topics, allowing the pupil and the teacher to concentrate on the principles of, or the relationship within the discipline. With the calculator, it is also possible to use realistic data, not numbers specially chosen to give a nice result. A further question raised by the use of calculators is that of the need of books of trigonometric tables now that calculators can produce the required trigonometric ratio of any angle at the press of a button. Are there, however, skills in the reading of tabulated values that we do not wish to lose? Perhaps such skills could be better taught in another setting, in, for example, timetables for public transport.

Algebra is a part of mathematics which, at first sight, may not seem to be much affected by the introduction of calculators and computers. The Mathematical Association in the United Kingdom has produced a booklet on that topic (1981). It is a short step from recording a calculation as

[3] [+] [8] [=]

to leaving one of the inputs open as in

[ ] [+] [8] [=]

so that the pupil can choose any key (and later any number, such as 2.5). Results can be tabulated. Pupils can be asked to describe what is happening, and already the calculator is contributing to the teaching of functions. A further activity is for pupil A to choose an input for the calculator (pressing the appropriate keys) and then to hand the calculator to pupil B. B then chooses a sequence of arithmetic operations, performs them and shows the result to pupil A. By choosing his inputs, pupil A has to discover the key sequence B has used. One important result of this kind of activity is that the pupil will move from his or her own actions towards expressions like

[2] [x] [ ] [+] [4] [=]   or 2 x [ ] + 4 =

and eventually to 2x + 4, rather than being presented with such expressions and required to work with them. In this connection, teachers should be prepared to spend time with expressions originated by pupils, such as '4 + 2x', 'x2 + 4', or '4 + x2', discussing their advantages and disadvantages, before moving to the commonly accepted form '2x + 4'. As well as giving meaning to elementary algebraic forms, calculators and computers also have major contributions to make to the teaching of topics such as graphs, manipulation and equations. These are discussed below.

Much of the discussion so far has obvious implications for teacher-training courses. The new material and the new approaches will have to be presented to student teachers, and they will need time to assimilate the ideas and to practise them. It is important that they should have the opportunity to work with pupils using calculators; the ease with which pupils accept the new technology can make teachers feel insecure. One approach could be to *require* each student-teacher to carry out some activities with a calculator (perhaps trying out one of the approaches above, or using one of the many calculator games) and to reflect and report on them in discussion with their fellow students.

New curriculum areas

If we consider the calculator or computer only in relation to the existing areas of the curriculum, we shall be making a grave mistake and losing much of the potential of the new technology. We need to think about the best ways of preparing our pupils for adult life in a society which will be extensively affected by microelectronics. Algorithms, procedures and modelling must be at the heart of the mathematics our pupils will do in schools.

Algorithms and procedures

A computer program implicitly defines an algorithm of some kind. So, pupils must be educated in the tasks of reading, performing, modifying (correcting, altering), designing and evaluating algorithms, Such work can be started with young pupils. It does not have to rely on sophisticated, computer-type languages. It can start with simple everyday language:

Choose a number.
Add 2 to it.
Multiply the result by 3.
State the result.

Later, pupils can introduce abbreviations as they feel the need for them. The important thing is for pupils to exchange ideas, with clarity of expression and flexibility of mind. Eventually, such work will lead to the use of structures encountered in recent programming languages, structures such as 'do – until', 'if – then – else' and 'while – do'.

Engel (1979) has written of the Klein reforms at the beginning of this century – the call for 'functional thinking' in school mathematics. He, in turn, calls for 'algorithmic thinking' and for the diffusion of such a way of thinking among teachers. He goes so far as to say: 'For every function "f" that we study in school, we should give at least one algorithm for its computation..... *without tears*' [Engel, 1979, p. 256].

In a large selection of examples, Engel includes those which are intrinsically interesting, such as [p. 259]

```
Start with a natural number n
while n <> 1 do
if n even then n ← n/2 else n ← 3n + 1
```
[1]

and new developments of standard pieces of the curriculum, such as [p. 261]

1. Note: the symbol 'n <> 1' means 'n < 1 or n > 1'; and the symbol '←' means 'is replaced by'; for example, when n is 5 we 'do' the 'else' and replace the 5 by 3 x 5 + 1, or 16.

> You put an amount x into a bank.
> At the end of each year the bank makes the replacement
> $x \leftarrow qx$, with $q = 1.05$ being a typical value,
> How long will it take until x is doubled, tripled, quadrupled . . .?

An algorithmic approach to mathematics has implications for the content of many areas of the subject in school. Among the important ideas which must be included are iteration, recursion, error analysis, sorting and searching. Many of these can be approached at a very simple level. Iteration can be introduced as trial and error which, with a fast calculating device, can be a speedy and powerful technique. Pupils may just 'try out' values of x in the equation $2.1x + 7.6 = 25$, to see how close they can get to the perfect 'fit'. Many will spontaneously tabulate their results and invent refinements of the method. Teachers will have to be sensitive to the need to guide some pupils towards an organized approach.

An extension of trial and error leads to a routine such as the method of bisection. This is based on a technique that is readily understood by most pupils. It assumes that if the graph of a function lies below the x-axis for one value of x and above the x-axis for another value of x, then it crosses the x-axis somewhere between these two x values. This assumption is valid for almost all the functions met in an elementary course. So, apart from the very few exceptions, we can take the value of x halfway between as being a good way to reduce the range of values, for it will give us a point where the graph is either above or below the axis (or, perhaps, exactly on the axis, in which case we need search no further) and we are back again at the situation we started with – a true iterative process. An example should make the method clear. Take the function:

$$f: x \rightarrow 4x^2 - 2x - 1$$

and tabulate its values like this:

x	f(x)	
0	–1	so, a crossing point lies between $x = 0$ and $x = 1$; take $x = 0.5$
1	1	
0.5	–1	so the crossing point lies between $x = 0.5$ and $x = 1$; take $x = 0.75$
0.75	0.25	so the 'solution' is between the values $x = 0.75$ and $x = 0.5$. Try $x = 0.63$ (there is no need to make it exactly halfway)
0.63	–0.6724	etc.

Much of this kind of work will be new to student-teachers, and they will need time (and encouragement) to become familiar with it. It has been observed, however, that pupils can take to such methods very quickly indeed. The implication of this is that it is quite likely that some pupils will investigate pieces of mathematics that the teacher has never met before. This means that the teacher's role has changed. Training courses must discourage student-teachers from thinking that they need to know all the answers.

Modelling

There have been increasing moves in some countries to introduce mathematical models more strongly into the curriculum. Advocates include Burghes (1982), Burkhardt (1981), Corcoran (1977) and The Spode Group (1981). We must start at an elementary level if we are to accustom pupils to the idea of a mathematical model. They must be made aware that when they carry out the calculation '4 x 3' to obtain '12', this is the extremely simple model of the practical task of calculating how many pebbles there are in a rectangular array with 3 pebbles along one side and 4 along the other. From an early stage, pupils must learn to perceive the movement from the physical world to the mathematical model and back again. This is the way to show how mathematics tackles real problems.

K. M. Hart (1981) describes the work of the research project 'Concepts in Secondary Mathematics and Science', an investigation which shed much interesting light on the ability of pupils in the United Kingdom to grasp the idea of a mathematical model. Pupils were asked to make up stories of situations which would lead to certain calculations (such as 9 x 3, or 84 ÷ 28) having to be performed. In almost all cases, the pupils found this kind of task more difficult than any of the 'problem' type of items in the tests used. Pupils were much affected by the size of the numbers involved, and they found multiplication and division more difficult than addition and subtraction, so that facility levels dropped to about 40 per cent with 12-year old pupils. The use of calculators did not feature in the work of this project. But the use of calculators was built in to some algebra texts produced in the United States (Corcoran et al, 1977), and modelling also made use of the calculator.

The following example is taken from the 1979 *Yearbook of the National Council of Teachers of Mathematics.* The information in Table 1 is first presented, for the American Buffalo:

Table 1. Data on the American Buffalo

	Current number	Survival rate	Yearly harvest
Adult males	M	0.95	H
Adult females	F	0.95	h
Male calves	m	0.50	none
Female calves	f	0.45	none

Then four equations are presented. First the predicted number of adult animals at the end of one year:

Males $A = 0.95M + 0.50m - H$
Females $E = 0.95F + 0.45f - h$

also the number of calves born during the year:

Males $C = 0.48E$
Females $D = 0.42E$

The pupils are then led, through a series of exercises which will make them familiar with ways in which these equations can be used, to consider the following problem:

The present population of the American Buffalo is about 26,000. Assume that 40 per cent are adult males, 35 per cent are adult females, 14 per cent are male calves and 11 per cent are female calves. Establish a constant harvesting policy that will maintain the population at about the same level over the next five years.

Work such as this is a realistic setting for equations and formulae. The shorthand is necessary, and the pupils are working on a problem of current interest. It should be pointed out that the example given is one of an early lesson in modelling, where the pupils are being led through the development of the model and then given an opportunity to investigate it. Later work would involve the actual development of models by the pupils themselves.

New methods of teaching and learning

It will be clear from what has gone before that new topics will appear in the school mathematics curriculum and that new approaches will be possible to topics already in the curriculum. Computers, in particular microcomputers, means that right across the curriculum new methods of teaching and learning mathematics will be possible, and there will be a greater variety of resources for the teacher to call upon. In this context, we shall briefly consider programming as a part of mathematics, the use of pre-prepared programs and packages and, finally, special-purpose computer languages.

The role of programming

Many mathematics teachers have been taught to program as part of their subject training, or have taught themselves to program, perhaps because of the strong mathematical content of programming. However, there have not yet been many instances of using programming as an integral part of pupils' mathematical learning. A number of examples do exist, (the reader might wish to explore the work of projects CAMP (Computer Assisted Mathematics Program) and SOLO in the United States) and the research associated with some of them (Hatfield and Kieren, 1972; Johnson and Harding, 1979) has produced positive results. These suggest that the writing of programs to study selected concepts enhances learning, and that pupils who study mathematics with computing are more apt to employ different problem-solving strategies effectively, and/or perform significantly better on mathematical problem-solving tasks. Although programming will arise naturally as a part of the study of algorithms, it is valuable to identify the particular contribution it can make to the learning of mathematics. Johnson maintains that it helps concept formation, concept demonstration and concept reinforcement. One example, taken from LaFrenz and Johnson (1969), will have to suffice to indicate one of the approaches taken. The reader may care to decide in which of the above three categories it should be placed.

The following program, when completed, will find the mean of 8 numbers. Study the program and answer questions a to e. Then complete the program.

```
 5  LET S = 0
10  READ N
20  FOR K = 1 TO N
30  READ X
40  LET S = S + ____
50  NEXT K
60  PRINT "MEAN FOR" N "SCORES IS" ____
70  DATA ____,12,9,13,7,11,9,14,10
99  END
```

a What is the purpose of the variable K?

b How many replacements are there for K? What are they?

c What are the replacements for N and X?
d How many times does the computer encounter statements 5 and 10?
e The loop in the program consists of which statements?

Almost certainly more than in any other subject, the small program, possibly as small as five to ten lines, has a major part to play in the mathematics classroom. It can be pupil- or teacher-devised, and pupils can be involved in its adaptation or correction, leading to important exchanges of ideas. The author has distinct memories of a group of 15-year-old pupils who produced the following program (written in BASIC) to investigate the behaviour of the gradient of a chord as the chord approaches the tangent to the curve (see Fig. 2).

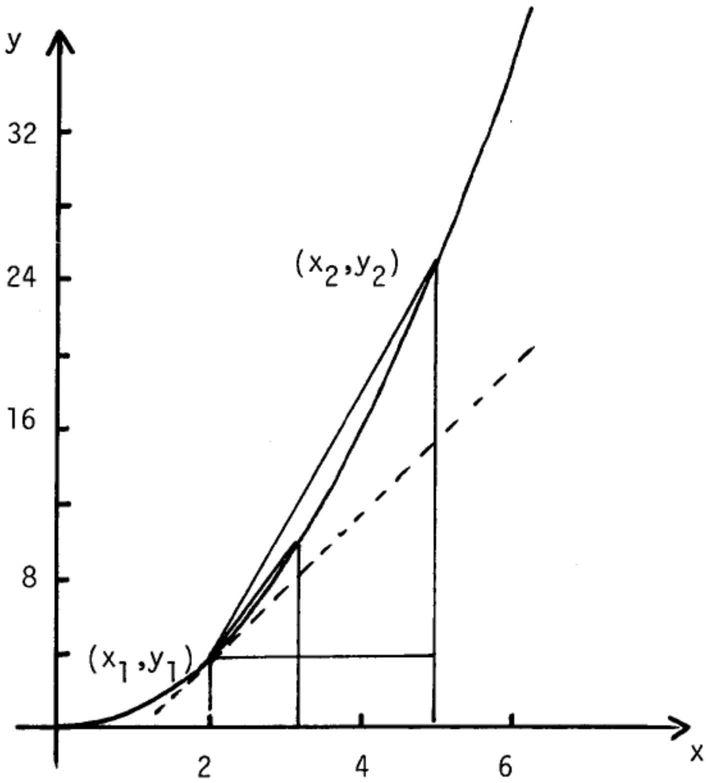


Figure 2.

The program printed out the gradient of each chord as the right-hand end of the chord approached the (fixed) left-hand end. It was as follows:

```
 10 INPUT X1                      (X1, Y1) is the left hand end
 20 LET Y1 = X1 ↑ 2               of the chord
 30 INPUT I                       I is the horizontal distance to
 40 PRINT X1, Y1                  the right hand end
 50 LET X2 = X1 + I
 60 LET Y2 = X2 ↑ 2
 70 LET G = (Y2 – Y1)/(X2 – X1)
 80 PRINT X2, Y2, G
 90 LET I = I/2                   this moves the right hand end
100 GOTO 50
110 END
```

For the starting point (2,4) the following values were printed out:

x_2	y_2	g
4	16	6
3	9	5
2.5	6.25	4.5
2.25	5.0625	4.25
2.125	4.51563	4.125
2.0625	4.25391	4.0625
2.03125	4.12598	4.03125
2.01563	4.06274	4.01563
2.00781	4.03131	4.00781
2.00391	4.01564	4.00391
2.00195	4.00782	4.00195
2.00098	4.00391	4.00098
2.00049	4.00195	4.00049
2.00024	4.00098	4.00024
2.00012	4.00049	4.00012
2.00006	4.00024	4.00006

at which point the program was stopped. There was already a wealth of material for the class to discuss. However, one pupil asked what would happen if the value given to I at the start were negative. The teacher was really left with no option but to try it – and the class had their first (and for the teacher unintended) demonstration of left-hand as well as right hand limits. The use of computers in the mathematics classroom will often produce the unexpected. We must train our teachers to be ready to face it and cope with it.

Computer-assisted learning packages

Increasingly, there will become available ready-written programs for the teacher to use – 'packages', often complete with material for pupil use as well as teachers' notes. Such packages may be for the teacher to use when teaching the class, or for groups of pupils to use, or even for the individual pupil to use. It is likely that the mathematics teacher will have at his or her disposal a set of 'utility' programs, which may be used time and time again at different stages of pupils' development. Some will plot graphs. Some will carry out routine statistical calculations, or produce the result of geometrical transformations. These packages will make increasing use of what is known as 'high-resolution graphics'. These are the detailed pictures or diagrams, in colour and even with animation which the computer can produce on a display screen. Surely such possibilities must make us think very hard about how we present mathematics to pupils in schools. Those who train for teaching must be encouraged to experiment with new approaches, and to discuss critically their advantages and disadvantages. The two examples which follow should indicate some of the new approaches.

A package produced by the Investigations on Teaching with Microcomputers as an Aid[1] (ITMA project, 1981), entitled 'JANE', is too large to describe in full here, but an example of its use may be informative. The program is intended for use with a wide range of pupils, both in age and ability. It is concerned with the idea of 'function' in mathematics, and it restricts itself to the operations of addition and subtraction. It allows the teacher to select a variety of functions and to choose alternative approaches to the investigation of each one. The presentation is very simple. An empty rectangle is shown on each side of the display screen, with either one or two figures of 'children' appearing in between. The idea of 'function' is then presented by assuming that a number which appears in the left-hand rectangle is changed in some way by the 'child' or the 'children'. The result can be displayed (if requested) in the right-hand rectangle. The problem is to discover what the 'child' or 'children' are doing with the numbers given to them. Fig. 3 shows a typical screen display.

1. College St Mark and St John, Plymouth, Devon, United Kingdom.

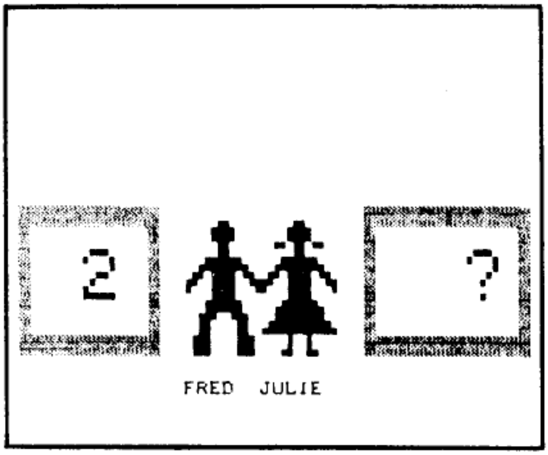


Figure 3.

Among the modes of operation of the package, the teacher can:

- allow the computer to choose the function;
- allow the pupils to suggest numbers for the right-hand rectangle, with the computer indicating if they have found the right result for that 'child';
- choose whether the 'child' is a boy or a girl (knowing, say, that girls multiply and boys add), or to have one of each;
- get the computer to give the result of the 'child's' action automatically;
- ask the computer to present the inverse problem.

The package has been used with children of various ages and abilities, and has provoked much discussion and involvement.

As a second example we take a program which has been used successfully by individual children and small groups. It has been produced by a group of teachers working in London, United Kingdom – SMILE (1983). The program is called 'VECTMEET'. On its display screen the pupil is presented with a 30 x 18 grid on which he or she can draw vectors, forming a continuous path. After each input – when the pupil has chosen a new vector – the computer also draws a vector. This too, is displayed on the screen. The computer chooses its vector by some secret rule by which it is related to the vector chosen by the pupil. The object of the game is for the pupil to make the two vectors meet. The computer chooses its rule for each game randomly from a set of ten. The ten include reflection in a horizontal or a vertical line, reflection in a diagonal line, enlargement with factor 2, 1/2, –2, –1/2, and

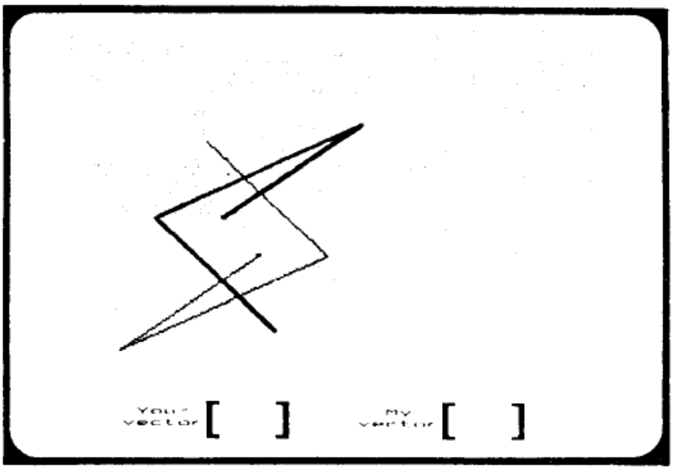


Figure 4.

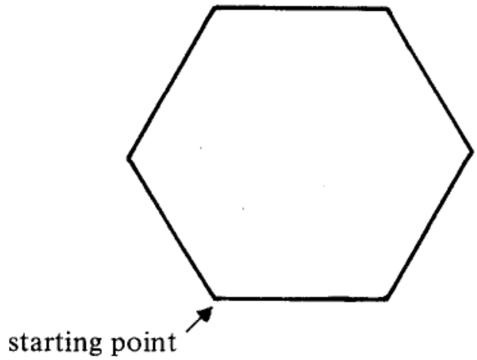
rotation through 90 or 180. Fig. 4 shows a typical screen display after the pupil has chosen three vectors and the computer has responded with three. This program is an excellent example of a game which motivates pupil activity, and one in which there is no need for a teacher to point out a pupil's mistake. It is immediately obvious if a pupil has not succeeded in 'catching' the computer's vector. The pupil is motivated to adapt his or her response, and to look for a strategy which will lead to success. Although, at first sight, the major activity may appear to be one of 'practice', there is much more involved. Observing, making a conjecture, generalizing all come in, and these are at the very heart of mathematical thinking.

Such a brief description of two programs cannot do justice to either of them, and certainly cannot give any indication of the range of programs available. It will certainly be necessary, as part of the education of mathematics teachers, that they should become familiar with some of the available material, and this will take time. There should also be opportunities for student teachers to try some of the materials with children, and to discuss the results with fellow students and their tutors.

Special-purpose computer languages

We are beginning to see the introduction of computer languages which are specifically intended for use by children learning particular subjects at school. Probably the most important example, at the moment, and one which has particular importance for mathematics, is LOGO, which was developed by Seymour Papert and his colleagues at the Massachusetts Institute of Technology.. It has its origins in the cognitive psychology of Piaget, and in ideas arising from the field of Artificial Intelligence. The whole approach starts from the assumption that children learn by doing, and by thinking about what they do. The language is simple, but powerful, and is usually introduced by getting the pupils to use the computer to control the movement of a 'turtle' – a small mechanical robot, which can be moved around on a flat surface and can leave a trace of its movements by lowering a pen onto the surface. The instructions given to the turtle (via the computer) consist of the movements the pupil wants the turtle to make. For example, to get the turtle to draw a hexagon the instructions might be:

```
FORWARD 50
TURN 60
FORWARD 50
TURN 60
FORWARD 50
TURN 60
FORWARD 50
TURN 60
FORWARD 50
TURN 60
FORWARD 50
TURN 60
```



The real power of the language, however, lies in the ways in which such programs can be shortened, and even linked together to create larger and more complicated drawings. The program above could be shortened (and given the name 'HEX') as follows:

```
TO HEX
REPEAT 6
     FORWARD 50
     TURN 60
END
```

Simply giving the computer the instruction 'HEX' would then produce the hexagon. The reader may care to write (in only five lines and using only the instructions met so far) a program which will react to the instruction 'STAR' and produce the drawing in Fig. 5.

Although work was started on LOGO many years ago, it is only recently that the language has been available on microcomputers, and thus used by many teachers. It is early days for the experiment, but the pace of development of the microelectronic revolution is so great that it will not take long for these developments to be widespread. The reader is urged not to take these brief remarks as indicative of the full power

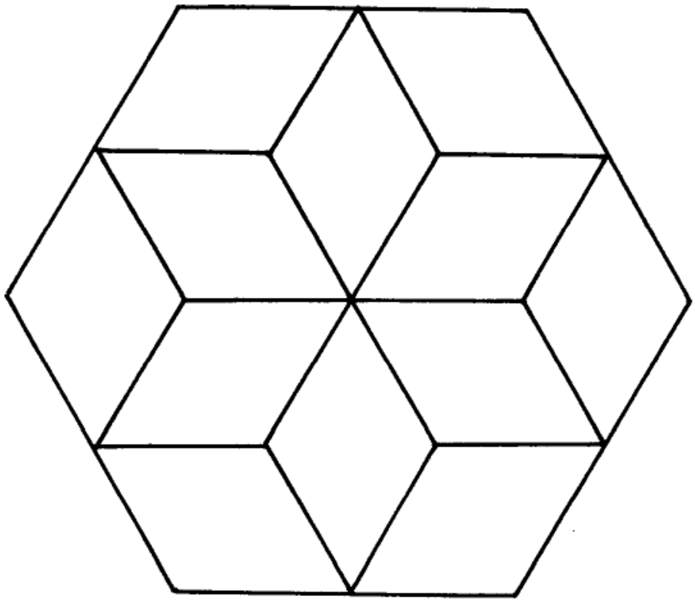


Figure 5.

of the language, but to read some of the reports of work done as, for example, those of Papert (1980), Howe et al (1979) and Papert et al (1979). The real importance of LOGO lies not simply in the creation of drawings but in the problem-solving processes developed by the pupils, the use of conceptual building blocks and the need to seek out the causes for the lack of desired outcomes and correct them – debugging as it is called.

Summary

It is inevitable that, in the presentation of a review such as this, (brief though it may be) there is an implicit presentation of particular view of mathematics and of the ways in which it should be taught or learned. Perhaps the best way to summarize what has been said is to make those views explicit.

In mathematics, we should be interested not just in the product, but in the process. We should educate our pupils so that they are able to use the mathematics they have learned to solve problems and to call on the appropriate tools to help them. The products of the micro-electronic revolution that we have seen so far will rapidly become available to more and more people and have an enormous potential for helping us to achieve these aims.

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Douglas A. Quadling

Algebra, analysis, geometry

The idea of a steadily developing mathematics curriculum is one to which we have grown accustomed over the past 25 years. We would do well to remind ourselves, however, that there are many to whom the association of the word 'change' with school mathematics seems strange, perhaps even contradictory. Parents are disappointed when their children have not learnt the skills which they themselves acquired at an equivalent age. Teachers of science subjects expect that their pupils will continue to have the same proficiency in carrying out the algebraic processes on which their own courses rely. Professors in the university assume that they can base their teaching on a foundation of mathematical knowledge which remains constant over time. Administrators oppose the introduction of calculators into the classroom for fear that the traditional emphasis on accuracy and facility will atrophy. Students training to be teachers imagine that their future role will be a mirror reflection of their own experience as pupils in school.

Even teachers who implement change in their own classes may be unclear about the reasons for it. Take for example the topic of number bases, which swept almost universally through the new curricula of the sixties. This was originally introduced in the belief that it would help children to a better understanding of numeration (and, in the case of binary notation, to an appreciation of the way in which computers carry out their function). The British School Mathematics Project (1965), in an early teachers' guide, stressed that:

> The purpose . . . is to revise the basic concepts and operations of arithmetic in a new context. It is not considered that the ability to calculate in bases other than 10 is important of itself.

But, within a short time, this topic came to be regarded as one of the cornerstones of 'modern' (and therefore, by implication, 'superior' and 'more advanced') mathematics, and much energy has been fruitlessly expended on developing such irrelevant computational skills.

So it is important to ask, why (and how) does the mathematics curriculum change.

Changes within mathematics

A first possible answer to that question is simply that mathematics itself changes:

> The major reform in school mathematics education which has been carried out in the USSR in recent years set out to remove a certain disproportion between the content of mathematics as an academic discipline taught in general schools and the content of mathematics as a science. (Firsov, 1983).
> In the course of the last half-century, the landscape of science as a whole, but especially the landscape of mathematics, has profoundly changed. Mathematics has reflected on itself, and its aims, its rigour, its power manifested in the extent and diversity of its applications have become radically different . . . We must prepare our children and students to understand and to use the mathematics of our own day. (Commission ministérielle sur l'enseignement des mathématiques, 1967).

It is, of course, rare for developments in mathematics at the research level to make a rapid penetration into the school curriculum. Those who, in the sixties, predicted that we should soon be teaching topology and Gödel's theorem in schools have been disappointed. There have been few disciples of Hilton (1975), who propounded the thesis:

> The spirit which led – or misled – me to recommend the fundamental group as a proper object of mathematical study in the high school leads me now to recommend some elementary categorical algebra in the high school mathematics syllabus as well . . . Certainly I would claim that the mathematics involved in elementary categorical algebra is very simple; it is bound to be by virtue of its universality . . . Categorical algebra codifies that very unit which any good mathematical syllabus must reveal and emphasize.

Similarly, the imaginative presentations by Frédérique Papy (1975) and Dienes (1978) of abstract mathematical structure to primary school children have remained individual *tours de force* rather than establishing new trends in mathematical education.

Indeed, the last few years have seen a move back from the more ambitious schemes of the previous decade, and a recognition that the incorporation of concepts from modern mathematics into the school curriculum requires a more subtle approach. As Willoughby (1983, p. 365) reported to the International Congress on Mathematical Education at Berkeley:

> There ought to be some strict limitations on the teaching of subject matter to ever younger children and these limitations ought to be applied long before we reach the obvious extreme of teaching prenatal ergodic theory. Even when teaching an exotic subject to a young child is possible, the question of whether it is desirable should be raised.

And Bingen (1982) captured the mood of the International Colloquium on Geometry Teaching at Mons when he referred to:

> the still more structured contraptions around vector spaces – about which everybody seems ashamed of, since they were barely mentioned.

This situation places teacher training in the front line of curriculum development. If new ideas from mathematics are to infiltrate school teaching, then it is the new entrants to the profession, coming fresh from their contact with current advanced mathematics at the university, who are best placed to introduce them. But the art of the teacher lies in distilling the essence of graduate school mathematics, and using this to enrich the diet of high school mathematics. Helping students to make this transition is the challenge for the teacher trainer.

Thus the teacher of secondary school geometry who has studied group theory in the university can be helped to see the links between the two. The school geometry course is peppered with cosets, conjugacy classes, normal subgroups and automorphisms for those who have eyes to see them. But the moral is not that we should teach group theory in school, but that the teacher should consider possible new interpretations and fresh emphases in his presentation of school geometry.

Again, most would agree that a first course in calculus is best not approached through non-standard analysis. Nevertheless, Schwarzenberger (1978) has suggested that, in the aftermath of that theory:

> (a) teachers need no longer be apologetic about using the Leibniz dy/dx notation, and can mention the idea of infinitesimals as part of its motivation, (b) each teacher should have ready a rough description of some model of the hyperreals (probably in terms of sequences of real numbers) which can be used to answer pupils' questions, (c) teachers should view with extreme scepticism patent treatments of the hyperreal numbers offered to them by over-dogmatic professional mathematicians, and (d) teachers must be particularly on their guard against the temptation to teach mathematics by intimidation as some brand of magic.

The possible effects of this development on the teaching of school calculus has been elaborated in greater detail by Tall (1980).

The implications for teacher training are well summed up by Vilenkin (1983, pp. 201-2):

> . . . the future teacher of mathematics is hardly likely to have to carry out research within the field of the mathematical sciences . . . At the same time his mathematical training must be sufficiently broad for him to be able to carry out his teaching not only in circumstances where syllabuses and textbooks are "stable" but also at times when various syllabus changes are taking place and when one textbook is being replaced by another which is more appropriate in the modern context. Mathematical

training of this sort must enable the teacher to make an independent assessment of the quality of various approaches to the teaching of mathematics in schools, to understand both the global aims of teaching and the nuances in the presentation of individual topics. The teacher must be able to teach both at a high level of rigour and by means of visual models, depending on the stage reached by his pupils.

By contrast, some new applications of mathematics can be quite rapidly and successfully incorporated into school courses. These, often calling for no more background knowledge and technique than is currently within the school mathematics currriculum, deal with issues with which the pupils find it easy to identify, such as traffic flow, distribution of goods and environmental improvement. Linear programming, for example, was brought (in an elementary form) into some British courses around 1965, and is now entering the curriculum for intending students of the social sciences in France and the Netherlands. Applications of graph theory and Boolean algebra are other topics which have been transplanted into the secondary school classroom.

The problem here for teacher training is to find ways of providing students with first-hand experience of using these topics in real situations. For if a teacher's own knowledge is based solely on book exercises, there is little chance that he will succeed in providing convincing motivation for his pupils. The ideal arrangement, which has been implemented successfully in a few instances, is to place students in relevant departments of industrial or commercial organizations, as a part of their academic or professional preparation, where they have opportunities to use their mathematics to tackle genuine problems. If this is not possible, then application projects can be devised to form part of the teacher-training course.

Experience of previous failures

A second reason for changing the mathematics curriculum is simply that the existing curriculum has been found deficient. Documentary evidence for this is understandably to be found in the writings of individuals rather than in official pronouncements. Nevertheless, it has to be recognized that the development of curricula remains an empirical exercise, and that the search for an ideal curriculum is modelled more appropriately upon an iterative process than upon an analytical solution.

It is important to appreciate that problems can arise at one of two levels of communication. First, the curriculum, as implemented in the classroom by teachers, may not match the intentions of those who devise it. Second, it may transpire that what is taught to the pupils is for one reason or another unsuitable.

At the first level, many of the difficulties experienced in implementing the new curricula in the seventies could be put down to attempting too much too quickly. In the United States, the report of the National Advisory Committee on Mathematical Education (1975) pointed out that:

When reports and conferences discuss what is 'current', meaning what is going on at the cutting edge of innovation, they sometimes give the impression that it is a time of great change and ferment. Meanwhile, back in the ordinary classroom . . . we may find little evidence of profound differences over a decade or so.

Nowhere was this more true than in the proposals for the reform of the algebra curriculum. Hirstein *et al* (1980) reported:

The emphasis shifted from algebra as a tool for solving real world problems to the presentation of an algebraic structure with its motivation in the future study as an abstract science. The old goal of immediate usefulness of algebra seemed not as important as the more mathematical goals of language precision and deductive justification. The new content and approach were sufficiently new to pose a threat to many of the teachers of mathematics in secondary schools.

Oldham (1980) wrote in similar vein:

It is perhaps sad that, some twenty years after the new ideas first began to affect Ireland, the 'modern' and 'traditional' parts of the algebra course remain very separate in so many pupils' minds . . . Without an understanding of the philosophical background, teachers may have difficulty in giving the courses a sense of direction. Without a thorough, internalised familiarity with the content, they may find it hard to devise suitably concrete teaching materials by which the structural concepts can be introduced.

And in the United Kingdom, the report of the Cockcroft Committee (1982) summarized the situation as follows:

Not all teachers possessed a sufficient mathematical background to enable them to appreciate the intentions underlying the new courses they were teaching. In consequence the material which was included in modern courses was often not presented as part of a unified structure but as a collection of disconnected topics whose relevance to the mathematics course as a whole did not become apparent to pupils.

There are important implications here for the training of mathematics teachers. Clearly it is essential that they should become competent in the mathematics they are required to teach; but this is not enough. Too often, courses of teacher training, both pre-service and in-service, have simply presented the new material as if it were the latest divine revelation, with no attempt to justify the change of approach or to relate it to the students' previous experience. Mathematics needs to be discussed as well as learnt, and nowhere is this more important than in the preparation of future teachers.

The trend towards abstraction, which was a feature of the first reforms, has also been found unacceptable to pupils. To quote again from Oldham (1980):

Whatever the virtues [of the courses] in academic terms, they do not appear relevant to many of the boys and girls who have no academic aspirations.

In the USSR, Pontrjagin (1980) attacked the irrelevance of the courses then existing on more general grounds:

The authors of the present day textbooks have taken an approach via set theory which is distinguished by its high level of abstraction and which assumes a certain level of mathematical culture which school children do not and cannot possess. Even the majority of teachers do not possess it. So what was the result? The artificial complication of material and excessive pupil workload, the introduction of formalism into the teaching and the distancing of it from real life and practice.

Certainly there seems to be little support for the view expressed by Hilton (1975) that 'It is surely common ground that students are not bothered by abstraction . . .'

The School Mathematics Project (1974) in the United Kingdom acknowledged, in response to external criticism, that in some respects the early reforms had gone too far: 'Looking back, we can see that the early SMP books were short of revision and drill exercises', an admission in marked contrast to the confident assertions (Thwaites, 1961) made at the outset of the project:

Greater emphasis should be placed on an appreciation of the structure of algebra – stressing the commutative, distributive and other similar properties – rather than on the acquisition of techniques. A greater understanding of the structure will inevitably enable this acquisition of techniques to be made.

Recent research, such as Concepts in Secondary Mathematics and Science (Hart, 1981), Strategies and Errors in Secondary Mathematics (Booth, 1981), and that of the Assessment of Performance Unit (Foxman, 1980) in the United Kingdom, has drawn attention to the difficulties experienced by pupils in attaching appropriate meaning to algebraic symbols. This research has important lessons for the mathematics curriculum and for teacher training, and points to the need for students to pay close attention to the interpretation of pupils' responses, both written and oral.

Not surprisingly, then, the current trend in syllabus revision is to draw back from abstract, deductive presentation of mathematical structures, and to concentrate rather on understanding and application at a less formal level. For example, in the United Kingdom the new syllabus for the School Mathematics Project at Advanced Level omits any explicit study of group theory, although a range of experience exhibiting this structure remains part of the course. A new curriculum for the penultimate year of secondary school for pupils not specializing in mathematics in French-speaking Belgium declares the objective to be 'familiarity with the principal functions from R to R', and goes on to spell out the significance of the word 'principal' as follows: 'that is to say, important for their role as elementary functions in mathematics, and for the part which they play in science and everyday life.'

In France, too, the curriculum for a comparable group of pupils warns against the complementary dangers of 'separating theoretical discussion from examples and applications', and 'reducing mathematics teaching to a jumble of bits and pieces'.

Changing views of mathematics

From what has been written so far, it might appear that the principal contemporary trend in mathematics teaching is the negative one – of a retreat from the ambitious programmes of twenty years ago to a position of neutral ground. There is, however, a more positive side to the debate, in the form of a major reassessment of the nature of 'mathematics' as a subject in the school curriculum.

This is strikingly put forward in a prologue to the new curriculum for the first year of the French lycée. Referring to the work in earlier classes, it states:

> The current programmes in mathematics for the first cycle were set up in opposition to a formalism which, riding roughshod over the intuitive background of the pupils, would isolate educational progress from the realities of experience. Underlying all good teaching is contact with the concrete world of the senses, stimulus for the pupil's personal activity, and elaboration of the means of investigation immediately applicable to the environment.

Going on to describe the next stage, it continues:

> The mathematics classroom is essentially a place of discovery, of exploring situations more or less controllable, of reflection on problems after they have been solved . . . Mathematical activity is not identified with the development of a well ordered sequence of theorems.

And in an earlier document (*Bulletin inter IREM*, 1981):

> Theory is not an end in itself, but a tool for providing answers to questions which arise in the real world.

In England and Wales, the Cockcroft Committee (1982, p. 71) pro-posed, as a fundamental requirement of good classroom practice, that:

Mathematics teaching at all levels should include opportunities for
exposition by the teacher;
discussion between teacher and pupils and between pupils themselves;
appropriate practical work;
consolidation and practice of fundamental skills and routines;
problem solving, including the application of mathematics to everyday situations;
investigational work.

The committee went on to elaborate these criteria:

It is often assumed that the need for practical activity ceases at the secondary stage, but this is not the case . . . Pupils of all levels of attainment can benefit from the opportunity for appropriate practical experience. The type of activity, the amount of repetition which is required will, of course, vary according to the needs and attainment of pupils.

(One might interpolate that the need for practical activity does not cease even on entry to university or to teacher training institution, or on becoming a professional mathematician or mathematics teacher.)

The idea of investigation is fundamental both to the study of mathematics itself and also to an understanding of the ways in which mathematics can be used to extend knowledge and to solve problems in very many fields . . . Investigations need be neither lengthy nor difficult. At the most fundamental level, and perhaps most frequently, they should start in response to pupils' questions.

The same sentiments emerged from a recent international meeting to discuss the teaching of geometry. Summarizing the proceedings, Bingen (1982) wrote that:

On one point, the agreement is impressive. While looking at geometry teaching, one has not to forget the learner. This has been the big mistake of the 'modern-math' programmes: too much thought on mathematics, not enough on children. And the learner one considers is not the old-day pupil that listens quietly to the teacher, but an active one. Learning geometry starts with manipulating objects, manually and intellectually.

The word 'intellectually' here is important. Practical activity is not an end in itself. As it proceeds, random experiment needs to be succeeded by selective experiment, by the evolution of a strategy, by recording results, by the formation of conjectures, and ultimately by reasoned justification. In mathematics one must move 'from the hands to the head'. To quote again from the French curriculum document: 'at the appropriate time, a stage of theoretical enquiry in depth is indispensable.'

Suppose, for example, that pupils are investigating the possible plane sections of a cube. The first steps might involve actual cutting of cubes formed out of potato, leading on to experiments in which attempts are made to fit cubes into holes of various shapes cut in stiff card. Various ideas may begin to form in the pupils' minds, such as these:

(1) the only possible shapes have 3,4,5 or 6 sides;
(2) the only possible triangles are acute-angled;
(3) the only possible quadrilaterals are trapezia, parallelograms, rectangles or squares;
(4) the only possible hexagons have their opposite sides parallel.

These conjectures still rest on a foundation of experimental observation. To become mathematical facts, they must now be justified by reasoning. For (1), we need to remark that two non-parallel planes meet in a line, that a cube has six plane faces, and that a plane polygon has at least three sides; (2) may involve reference to the theorem of Pythagoras; (3) and (4) require the observation that parallel planes are cut by a transversal plane in parallel lines. It is clear that, to direct a lesson along these lines, the teacher needs to be flexible, to exercise judgement, and to know when to prod and when to remain silent. The model teacher described in the French document:

> must have a deep appreciation of the topic he is teaching, and be able to expound it clearly; but the ideal is not to engage the pupils in a dialogue at this level of perfection, but to train them, when faced with situations as they arise naturally, in the habit of reflection and the exercise of personal initiative.

The teacher envisaged by the Cockcroft Committee has similar qualities. He

> must be willing to pursue the matter when a pupil asks 'could we have done the same thing with three other numbers?' or 'what would happen if . . . ?' . . . The essential requirement is . . . that the teacher takes the opportunities which are presented by the members of the class. There should be willingness on the part of the teacher to follow some false trails and not to say at the outset that the trail leads nowhere. Nor should an interesting line of thought be curtailed because 'there is no time' or because 'it is not in the syllabus'.

Obviously this is not the kind of teacher likely to emerge from a course of training in which the instruction is entirely expository. If we expect teachers to adopt these methods in their classes, then they need to experience this approach to mathematics at their own level, and its importance needs to be emphasized through changes in the forms of assessment. For example, in the United Kingdom, on national in-service qualification for teachers, the Diploma in Mathematical Education of the Mathematical Association, requires every teacher to submit a folder of personal investigations as one of the four components contributing to the award.

New technology

It is a fortunate coincidence that this shift to an active, exploratory view of mathematics teaching should come at the same time as the arrival on the scene of the electronic calculator and the microcomputer – surely the most significant new development for mathematical education of the present decade. Not only do these facilities raise questions about the teaching of all the traditional subject matter, but they open up possibilities for completely new interpretations of the mathematics curriculum.

Of course, the timing and pace of development of the microelectronics revolution vary widely. Already, in some countries, every pupil in a senior secondary school class will have his or her own calculator, programmed with an extensive range of algebraic, trigonometric and exponential functions and possibly, even, with procedures for numerical integration and statistical calculations. Moreover, there may be several microcomputers in the school to which pupils and teachers have easy access, and one or more of these may be located in a mathematics classroom. In this situation, the use of slide rules, printed books of tables and perhaps even graph paper is almost obsolete, and it is realistic to be rewriting the curriculum to take account of these facilities. But to other readers of this chapter, such a state of affairs exists, for the time being, only in dreams.

In the context of teacher training, however, it is already urgent to consider the implications of the new technology, whether or not it is an immediate reality in the classroom.

Consider, as an example, the topic of curve sketching in analysis, which has traditionally been accorded a high priority in the upper classes of the secondary school. There can be no doubt that this can provide an excellent training for pupils as an exercise in organizing a sequence of calculations and applying them systematically to achieve a specific end, and checking one piece of evidence against another in the process. But where is the motivation for it when an outline of the graph of any function can be called up on the screen at the touch of a few keys? We have to ask afresh, what is the real purpose of this work. If it is simply that we need to know the shape of the graph, then the computer will provide this to almost any level of detail which we care to specify. If, on the other hand, its place in the curriculum rests on its value as a discipline, are we not risking the charge of irrelevance?

And this is not an isolated example. The numerical calculation of definite integrals, the solution of differential equations for a variety of boundary conditions, and the numerical manipulation of polynomials and rational forms are equally called into question. Should we continue to teach them as if the technology did not exist?

It has even been suggested that, since the operation of computers is essentially discrete, we ought no longer to teach infinitesimal calculus, but concentrate instead on finite difference techniques; but this is not a view which yet commands general support. It can be argued on the other side that once the conceptual jump to limiting processes has been made, infinitesimal calculus is less complicated technically than finite difference calculus. Continuous models based on the real numbers are in general found simpler to handle than discrete models based on a subset of the rationals. So the tendency at the present time is to present the theory in infinitesimal form, even if, ultimately, its application will involve translation into a discrete process. However, this is a point which will need to be borne in mind as computer applications assume an increasingly important role in mathematics at all levels. (See Roberts, 1983; Webb, 1983)

Meanwhile, we should be alerting ourselves to the positive gains to be achieved from teaching mathematics using the computer as a resource. The facility of being able to examine in quick succession the graphs of a family of curves with various values of a parameter offers scope for imaginative experimental work (either by an individual pupil, or with a small group or with a whole class) which could lead to a number of conjectures and subsequent analytical investigation. Many maximum and minimum problems can be explored empirically before pupils are familiar with differentiation, and this provides concrete experience and motivation for theoretical investigation, using calculus at a later stage. A computer program for rotation and translation in three dimensions makes it possible to display on the screen the various plane sections of a given solid figure (as in the example of the cube described above). These are just a few examples of the new possibilities opened up by introducing a microcomputer into the classroom; it is certain that this will be an area of vigorous curriculum development in the next few years.

It is the combination of almost instantaneous response with high quality graphical output which captures the attention of pupils and makes the microcomputer such a powerful tool in teaching geometry and analysis. The following illustration, based on the writer's personal observation, gives an indication of the possibilities. In a class of 13-year-old children in a school in Belgium, the teacher had prepared a program which would display a point, an arrow, or one of a number of different shapes on the screen in any position; this could then be rotated through any chosen angle about any point of the screen (with the option of showing also the path traced out by the vertices). In the course of one lesson the teacher was able to begin by building up the pupils' awareness of the effect of the rotation operation – showing, for example, that keeping the angle constant, but varying the centre of rotation, changed the position of the image but not its orientation. Next, two copies of the shape were shown on the screen with different orientations, and the pupils were invited to locate the centre and angle of the rotation which would map one to the other; their answers could then be checked by showing the pupils' suggested rotation performed. Initially they worked largely by guessing, but as the lesson progressed the advantage of devising a theoretical solution became apparent. The lesson remained within the control of the teacher, but throughout the pupils were actively engaged in trying to solve the problems presented. It would have been necessary to follow up this lesson with drawing exercises, and later with reinforcement of the theoretical discussion; but in this instance the use of the computer greatly enriched the learning experience. Moreover, there was the possibility for the teacher at a later stage to recall the program, to revise or extend the work either with the whole class or with a selected group, or even for the pupils to carry out their own experiments.

Even the hand-held electronic calculator can become far more than a machine for computation, and can be used to powerful effect in building up mathematical ideas, particularly in algebra and analysis. For example, using a very simple calculator with just four arithmetic operations, attempts to find a number r such that rxr = 2 may lead to the successive location of r within the nested intervals [1,2], [1.4,1.5], [1.41,1.42], [1.414,1.415], and so on – a concept of fundamental importance in assigning a meaning to the real number $\sqrt{2}$. Again, with a more advanced calculator, the investigation of limits can be greatly enriched. Thus a sequence such as $\sqrt[n]{n}$ can be evaluated numerically for several increasingly large values of the variable n:

n	$\sqrt[n]{n}$
10	1.2589 . . .
100	1.0471 . . .
1000	1.0069 . . .
10000	1.0009 . . .

These computations suggest strongly that the sequence has a limit, namely 1. What is more, they invite questions such as: 'how large would n have to be before $\sqrt[n]{n}$ becomes less than 1.0001?', a question which takes one a large step towards the definition of limit. The pupils can then use their calculators to answer this question by a 'trial and error' strategy, finding that n = 116679 (or any larger number) will achieve this. In this way the ground is prepared conceptually for the question:

'how large would n have to be before $\sqrt[n]{n}$ becomes less than $1 + \epsilon$?', and the proof that the limit is indeed 1.

How can teachers be prepared to make use of these new technological possibilities? In the first instance, they must of course have the opportunity to become thoroughly familiar with the hardware themselves – not just as passive observers or to carry out set exercises, but to investigate mathematical questions which are of interest to themselves. This implies that they should have easy personal access to calculators and microcomputers for extensive periods of time.

Second, the students must have opportunities to observe children at work with the equipment, to listen to their discussions and to assess for themselves the implications for their own teaching. It is important for them to realize that the pupils need to use calculators and computers for themselves, to be able to investigate problems in their own way, and that their role as teachers is to guide the pupils' enquiries rather than to demonstrate ready-made solutions.

Lastly, there has to be a change in mathematical stance. It is likely that most students who are at present training to be teachers will have had a far more formal, algebraic education in mathematics than is like the one outlined in the preceding paragraphs. Considerable imagination will be needed if they are to appreciate the possibilities opened up by these new advances and the further technological developments which are certain to come in the next few years.

A report recently published in the United Kingdom (Fletcher, 1983), based on visits to schools which are already using microcomputers in the teaching of mathematics, summarizes the situation as follows:

> It should be part of the equipment of every mathematics teacher to be able to construct [short] programs on the spot – but it is just as important to develop in pupils the power to do so. There are programs of this kind relating to almost any part of traditional algebra and arithmetic, and to many parts of geometry as well . . . A major phenomenon which is taking place in many schools at the moment [is] the astonishing response of the pupils to computers if they are allowed to use them in their own way and to decide on directions for themselves . . . The outstanding implication for the teaching of mathematics is that this energy should be developed. The pupils may not have recognised the mathematical aspects of what they are doing: it is for schools to generate direction and continuity from these spontaneous beginnings.

The special problem of geometry

'The purpose of the geometry [in the high school curriculum] is to teach the student something about precise and deductive reasoning.' (Herstein, 1975)

'Geometry is endangered by dogmatic ideas on mathematical rigour. They express themselves in two different ways: absorbing geometry in a system of mathematics as linear algebra, or strangulating it by rigid axiomatics. So it is not one devil menacing geometry . . . There are two. The escape that is left is the deep sea. It is a safe escape if you have learned swimming. In fact, that is the way geometry should be taught, just like swimming.' (Freudenthal, 1971)

For many years school geometry has lacked a clear sense of direction. Euclid has gone, but, in the vacuum created by his departure, there is chaos. Summarizing the recent International Colloquim on Geometry Teaching, Bingen (1982) reported that: 'No single concept of the geometry one should teach at secondary level has a majority of persons behind it.'

He went on to add a warning: 'This is probably the chief reason why geometry has so much difficulty resisting to the pressure of the competing branches of mathematics in high school curricula.'

And this is a matter of concern: some delegates to the colloquium reported that teachers themselves were giving low priority to geometry as against algebra, and that the root cause of this lay in the mismatch between teacher-training courses and the needs of pupils in school.

This, perhaps, is not so important if one's view of geometry accepts at its face value the statement of Herstein quoted above, for it may be possible to provide an equivalent training in 'precise and deductive reasoning' through algebra, or even through computer programming. But most practitioners and users of mathematics regard geometry as occupying a central role in the subject, and there is general agreement that geometry needs to be rescued from its present malaise.

In countries which have moved away from teaching geometry as a deductive system, there are some which put great emphasis on the process of systemization, by reflecting on perceptions of physical space, allowing the theoretical models (with their definitions and theorems) to emerge through the activity of the pupils (de Lange, 1982). Freudenthal (1971) has described the work of the van Hieles and van Albada, designed 'to stimulate a geometric activity of folding, cutting, glueing, drawing, painting, measuring, fitting, paving'. Others have laid more emphasis on the process of classification using the concepts of group theory (Buekenhout, 1980), providing children with a more structured environment within which they can identify groups of transformations, possibly going on to link these with algebraic representations.

Other countries, however, have continued to emphasize geometry as a deductive system, and the thrust of their reform has then been directed towards finding alternative sets of axioms which can be more successfully implemented at school level. For example, Teslenko and Firsov (1983), describing the new geometry course in the USSR for grades 6 to 10, write:

The system of axioms selected by Pogorelov makes it possible to construct the whole basic core of the content of the geometry course – the totality of geometric facts – in a strictly deductive manner. Proof reaches a high level. Thus for example many properties of motion (displacement) which in the existing textbook are accepted without proof are here proved.

And when transformation geometry came to the United States, the Euclidean forms of argument and standards of rigour were retained, even though the material was substantially changed. (Coxford and Usiskin, 1971)

There has been little guidance from the universities. Whereas a school course in analysis can be seen to be developed and extended at the next level, geometry has all but disappeared from many university courses. In the University of Cambridge, for example, in 1983–4 only one short, non-examinable course in geometry and one course in algebraic geometry were available to undergraduate students (although clearly much that is geometrical in conception features in other courses). Where such courses do exist, they may be of little help to school mathematics:

University teachers give very formal and thorough courses in (for example) transformation geometry, and the young teachers then try to teach the subject in the same way to their school classes, simply replacing one monolithic structure by another. Whilst it is in general desirable for teachers to understand a subject to a greater depth then they will teach it, they also need to be helped (perhaps by means of workshop-type activities) to make the transition from the university lecture-room to the school classroom. (Quadling, 1982)

Revuz (1971) states the dilemma dramatically:

Is it sensible to consider today that there exists a relatively independent part of mathematics that is called geometry? As a mathematician, I will certainly answer: No! But, if I am asked if geometry must be taught, I feel compelled to say: Yes!

The situation seems to have clarified very little since those words were written. Bingen (1982) concludes: 'Without a clear sight of the end of the journey – whatever it is – geometry teaching is in danger of becoming a random walk.'

For the teacher trainers, then, it would seem that many options remain open. Perhaps it is less important than we have thought to be agreed on precisely what geometry we teach than that we teach *some* geometry, and that we do so in a spirit of open enquiry rather than a didactic formalism. The challenge is to devise courses of teacher training which encourage students to get 'glue on their fingers' whilst, at the same time, matching up to the academic expectations of the university in terms of intellectual demands.

Conclusion

Despite these uncertainties about geometry, the general trend of mathematical education is clear. Twenty years ago it was correctly diagnosed that there was a sickness in school mathematics: children were being educated for a world which no longer existed. The physicians were called in, and a régime of mathematical rigour was prescribed. Children must be placed in an antiseptic environment and dosed regularly with mathematical structures. For a time, the patient seemed to rally; but it was not long before rejection symptoms were observed. Abstraction was not a hormone which could be imposed from outside, but one that the patient must generate for himself in response to appropriate stimulation. What he needs is not isolation from the external world, but fresh air and exercise. Mathematical health is to be found not in the contemplation of ideal systems, but in the rough-and-tumble of active participation and creation.

Now we have to educate the nurses to put this remedy into effect: not to shut the windows and give the patients their regular dosage, but to lead them out into the open air. It is a task which calls for far more imagination and personal judgement, but one which hopefully can offer correspondingly greater satisfaction.

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Lennart Råde

Statistics

Introduction

The content of the mathematics curriculum at the school level has, in most countries of the world, been subject to constant debate (and reform) during at least three decades. Numerous recommendations for changes have been formulated at national and international conferences, and by national and international working groups. One very popular recommendation has been that probability and statistics be included in the curriculum. Recommendations have been severally formulated by:

a) The Commission on Mathematics organized in the United States in 1955:

Just as mathematics deals with situations in which the facts can be determined, it also provides ways to study, understand, and control uncertainty. Many of the newer applications of mathematics use the theories of probability and statistical reasoning.

Increasingly, modern science – physics, biology, social science – makes use of probabilistic descriptions of phenomena.

The Commission believes that it is desirable that material in these areas be introduced into the high school curriculum. Statistical thinking is playing more and more of a part in the daily lives of educated men and women. An introduction to statistical thinking is an important supplement to an introduction to deductive thinking. (*Program for College Preparatory Mathematics; Report of the Commission on Mathematics,* 1959.)

b) The Royaumont Seminar (the famous seminar on mathematics teaching organized by the Organisation for European Economic Co-operation (OEEC)[1] in 1959):

1. The Organisation for Economic Co-operation and Development (OECD) since 1961.

Elementary probability must be recognized as an appropriate part of the mathematics taught in secondary schools.
a) Statistical inference must be recognized as applied mathematics which contributes in an essential way to decision processes in the spirit of the 'scientific method' basic to so many fields, both in the physical sciences and in the science of human behaviour. Furthermore, it must be recognized that statistical reasoning is of growing importance in the field of public affairs.
b) Suitable elementary instruction in probability and statistics must be introduced into the curricula of secondary schools.
c) Suitable preparatory courses for teachers of these subjects must be introduced in normal schools and teacher-training institutions. (*New Thinking in School Mathematics,* 1961, p. 122.)

c) The OECD-sponsored Conference on Mathematics Education held in Athens in 1963:

It is necessary to recognize the importance for science specialists of the following topics:
vector spaces, the calculus, and probability and statistics.

Also, students other than science students should receive a sound mathematical education. Their courses should include the fundamental concepts, together with a knowledge of their applications. Particularly, these courses should include probability and statistics. (*Mathematics To-day; A Guide for Teachers,* 1964, p. 305.)

d) The Cambridge Conference on School Mathematics (United States), also in 1963:

We suggest that probability be taught in four doses through the curriculum.
1. In the elementary school, empirical study of the statistics of repeated chance events, coupled with some arithmetic study of the workings of the law of large numbers.
2. In junior high school, probability as an additive set function on finite sets. Conditional probability, independence, binomial distribution, expectation, variance, and some simple statistical tests.
3. In senior high school, after the first work on limits and series, probability as an additive set function on countable sets. Poisson distribution, law of large numbers, etc.
4. In senior high school, after integral calculus, probability as an additive set function of intervals on the line. Continuous distributions on the line and in several dimensions, normal distribution, limit theorems, etc. (*Goals for School Mathematics; The Report of the Cambridge Conference on School Mathematics,* 1963, p. 71.).

One would think that, with such support, probability and statistics should by now have a well-established place in school curriculum in most countries of the world. This seems, however, not to be the case. The state of statistics teaching in schools throughout the world has recently been surveyed by the task force of the International Statistical

Institute (ISI) on teaching statistics at the school level (Barnett, 1982). This survey consists of reports from different countries of the world, and they very often express dissatisfaction with the present state of statistics teaching. For example:

It is a particularly serious matter that in the scientific sector of the academic schools stochastics is completely ignored. (A. Zuliani, Italy)

Although probability theory is now considered by mathematicians as belonging entirely to mathematics, and, although most subjects use it, its teaching in France is discredited in the eyes of most mathematics teachers. It is dealt with separately, if time is left, or if it is required for an examination, and it is the first topic to be omitted in any syllabus reduction. (P. L. Hennequin, France)

There are at present no administration or independent curriculum development projects on teaching stochastics in the Federal Republic [of Germany]. There is no agreement, not even a serious discussion, on which parts of the syllabuses (for the 10 – 16 years old) should be cut in favour of material from stochastics.
(H. Dinges, Federal Republic of Germany)

Up to this date, we have no independent curriculum of teaching statistics in school education in Japan. (Y. Ukita, Japan)

In some countries, however, the situation is somewhat better. In Hungary, for example, as reported by T. Nemetz (Nemetz, 1982), 'probability and statistics' has been in and out of the curriculum since 1849. At present, these subjects are prescribed for all grades 1 to 8 of the primary school. In the secondary school (*gymnasium*), however, 'probability and statistics' has been absent since 1973. In Sweden, probability and statistics is treated to different extent in all options of the secondary school, as described by A. af Ekenstam (Ekenstam, af, 1982). Here, too, a new mathematics curriculum for the 'economics, special science and humanistics' option will start in 1984. This will include a course in statistics, which in specific content will be the largest area of the curriculum. In England and Wales, most school curricula in mathematics contain probability and statistics. Of special interest was the curriculum development project 'Statistical Education' sponsored by the Schools Council and based at the University of Sheffield during 1975 to 1982. This Project produced, tested and evaluated a large number of teaching units in statistics (Holmes, 1980).

Various circumstances may be identified to explain the slow introduction of statistics into schools. Here are some:

The foundations of statistics as a scientific discipline are under debate. There are different opinions about how to define 'statistics'. There are also controversies about fundamental matters, for example, the controversy about Bayesian statistics. For a comprehensive discussion of the foundations of statistics, see Barnett (1973), and Lindley (1970).

What should be the place of statistics in the school curriculum? Should statistics be treated as a separate subject, or as part of mathematics, or should it be introduced with an interdisciplinary approach?

Very little is known about the didactics of statistics. We can, however, probably expect much work and research in this field. The First International Conference on Teaching Statistics held in Sheffield, United Kingdom, in August, 1982 (with about 500 participants from about 50 countries), reflects a concrete indication of the interest in the teaching of statistics. Some books like Kütting (1981) and Shulte and Smart (1981), and the journal *Teaching Statistics* (University of Sheffield) are other indications of this interest.

There is in most countries a shortage of teachers who are qualified to teach statistics (because their training did not include any statistics). There is also a lack of appropriate teaching material.

Probability, statistics or stochastics?

These terms deserve some comment. They are often used in discussions of curriculum. Usually, 'probability' and 'statistics' are considered to be different but closely-related subjects. (For people with a non-frequentist, or Bayesian, attitude, however, there is no real distinction between the two subjects.) Probability can be considered a mathematical subject, which, however, has application in many different fields. One such application is in statistics. What 'statistics' is has been discussed and analysed (Barnett, 1973). Barnett summarizes his analysis of different definitions of 'statistics' in the following general definition: '. . . statistics is the study of how information should be employed to reflect on, and give guidance for action in, a practical situation involving uncertainty.'

Two key words in this definition are 'information' and 'uncertainty'. Instead of 'information', the term 'data' could be used.

In some countries, especially in German-speaking ones, the term 'stochastics' is used, particularly in curriculum discussions. For example, in 1980, a conference was arranged in Klagenfurt, Austria, having as its theme 'Stochastik im Schulunterricht' [Stochastics at School Level] (Dörfler and Fischer, 1981). By 'stochastics' is probably meant probability theory and all its applications, especially statistics. It has been pointed out that a danger with this term is that statistics is not given its appropriate place in the curriculum.

Why teach statistics at school level?

In the recommendations which have been cited above, several different reasons have been given for teaching statistics at the school level. Holmes (1981, *a, b*) gives a full discussion and analysis of this issue. So does Barnett (1983). Summarizing his analysis, Holmes gives five reasons for introducing statistics at the school level. They are: statistics is an integral part of our culture; statistical thinking is an essential part of numeracy; exposure to real date can aid personal development; statistical ideas are widely used at work after school; and an early exposure to statistical ideas can give a basic intuitive understanding of the subject.

Trends in statistics education at school level

We will now try to identify some characteristic trends in statistics education at school level. They are not based so much on established curricula as on suggested reforms and ideas. Our analysis is based mainly on a survey (Barnett, 1982) and on conference reports (Dörfler and Fischer, 1981; Grey et al., 1983). The summary (Borovcnik, 1981) has been particularly helpful. So has Eicker (1983). There appear to be five main trends: more emphasis on statistics than on probability, with special emphasis on descriptive statistics; emphasis on applications and model building; use of simulation; use of calculators and computers; and use of project work.

Emphasis on statistics, especially descriptive statistics

In many countries, the introduction of statistics at the school level has resulted in a course in probability with, perhaps, a short treatment of some statistical techniques, such as the estimation of the two parameters of the normal distribution. There are two reasons for this. One is that it is usually intended to base statistics on probability theory. So, to do statistics, it is necessary to lay down a substantial foundation in probability. A second reason is that probability theory is closer to traditional mathematics, and teachers at school level feel more at home with probability than with statistics. It is, however, possible to offer a reasonable course in statistics without a substantial probability background, as the probabilistic concepts can be introduced as and when they are needed. And, as a matter of fact, a great deal of statistics can be taught without a background in probability. This is discussed at length by Winter (1983), who gives, and argues for, five reasons for teaching descriptive statistics. They are: descriptive statistics is more substantial than many people assume; teaching descriptive statistics fosters the general aims of education in compulsory schools; the methods of, and the pursuits in descriptive statistics may concern, probably more than other mathematical activities, the pupil's actual existence; descriptive statistics can easily be merged into the canonical syllabus without overloading it; and, in order to benefit from the education value of descriptive statistics, the mathematical modelling of problems in the pupils' surroundings should be one of the main and fundamental activities.

In connection with the teaching of descriptive statistics, the possibility of introducing methods from exploratory data analysis (EDA) should be mentioned. EDA provides a number of very effective, mainly graphical, statistical methods developed by John W. Tukey. Koopmans (1981) and Velleman and Hoaglin (1981) are examples of statistical textbooks making an integrated use of EDA. The pro's and con's of EDA at the school level are discussed by Bibby (1983).

Emphasis on applications and model building

Traditionally, games of chance have played an important role in teaching stochastics. Moreover, many important results and methods in stochastics have their origin in games of chance. But the students might not get an appropriate understanding of the importance of stochastics if the only examples they meet are about gambling. So a trend in statistics teaching today is to illustrate the subject with concrete applications from fields like science, technology, insurance, demography, traffic control, social science, administration and so on. A serious problem, however, is to find appropriate applications from these wider fields. Some collections of such examples have been published, for example Mosteller et al. (1975) and Gubbins et al. (1982). Examples can also be found in Tanur et al. (1972) and in the journal *Teaching Statistics* (University of Sheffield).

The choice of applications is important also from another point of view. It will broaden the students' experience of mathematical model-building, especially the making of non-deterministic models. The emphasis on mathematical model-building is an important trend in teaching today, not only in statistics but also in mathematics generally.

Use of simulation

Simulation is an important tool, and an important principle in stochastics. It can be used to study random experiments where an analytic treatment is not possible. But in teaching at the school level it can also be used as a didactic tool. For example, it can demonstrate, in a concrete and challenging way, both the variability and the stability in a random experiment, or it can motivate the introduction of fundamental concepts like probabilities and expectations. Simulation can also be used to study estimation procedures in statistics and other statistical methods.

To use simulation effectively in statistics teaching at the school, the students should have access to programmable machines, such as programmable calculators or computers. The use of simulation in teaching stochastics at the school level is discussed in Råde (1976; 1981 *b;* 1983) and G. Schrage (1981).

Use of calculators and computers

Computers and programmable calculators are important tools in statistics. This is one good reason for using them in teaching statistics at school level. But a more important reason is that they can be used as a didactic tool, to give the teaching of statistics a completely new dimension. Computers can be used to make simulations, as discussed above, for handling large sets of data and for the calculation of probability distributions. The importance of computers has been known for quite some time; it is remarkable that so very little has been done to employ them in schools. Very little is said about calculators and computers in the reports from different countries about the teaching of statistics at the school level in Barnett's survey (1982). The Swedish ARK-project is the only experimental programme mentioned which aims to integrate computers into the teaching of statistics. In this experimental programme (Råde, 1981 *a*), the written teaching material requires programmable calculators and computers to be used from the very beginning. The programme has had a great influence on the teaching of statistics at school level in Sweden.

Use of project work

The use of project work has been discussed at length by Holmes (1980). He gives the following reasons for including project work in the teaching of statistics: they put the use of statistical techniques into a particular context, and bring out their relevance; they are more motivating to learning statistics than routine lessons are (this is especially so if the student can select his own topic from one of his own areas of interest); they give a faster feel for real data, its accuracy or otherwise, its essential variability, and the degree of reliability that can be placed on conclusions (they often raise problems of measurability – is it possible to measure what we want to measure or do we have to use instead some other indicator?); they emphasize the applications of statistics and bring out the value of learning statistical techniques; and they show that statistics is more than a branch of mathematics, and they put into their proper perspective the usefulness of mathematical techniques.

It should be realized that the use of computers facilitates challenging project work. Several such projects have been worked out within the framework of the Swedish ARK-project. One of them is fully described by A. af Ekenstam (1981).

Implications for teacher training

The foregoing has certain implications for teacher training. In this discussion of these implications, the assumption is made that statistics teaching in schools will become part of the mathematics curriculum. This implies that future teachers of mathematics have to be prepared to teach statistics. Consequently, it is now imperative that statistics, including probability, be given an appropriate place in the teacher education programme of universities and teacher-training colleges. This means that between 15 and 25 per cent of the subject-oriented courses should be devoted to probability and statistics. This, it is important to note, does not imply an equivalent loss of mathematics courses since the statistics courses will provide the students with opportunities to use mathematics in an applied context. The statistics courses should also be specific courses for teacher students, and adapted to the special needs and interests of the students.

Teacher education should include the didactics of statistics. Traditional teacher education programmes do, of course, include courses on the didactics of mathematics. Parts of these courses will be relevant to the teaching of statistics, but particular aspects and the peculiar problems of teaching statistics will not be covered. In this respect, there is a great need for teaching material relevant to the didactics of statistics, and it is to be hoped that the modules for teacher training, which the Unesco Statistical Education Committee is producing, will help to fill the void.

Teacher education should include work with computers. Computers will be used more and more at the school level, and the future teacher must be prepared to employ computers in teaching statistics. For a number of years, the author has given a course at the University of Gothenburg entitled 'Statistics for Teachers'. In this course, the students are required to devise some units of work with computers and programmable calculators. The units include the generation of random digits, the test of random digit generators using frequency and poker tests, simulation, and work with specific distributions.

Teacher education should include mathematical model-building of situations with elements of uncertainty. The process of mathematical model-building is a complex one. It can be analysed according to different schemes, including assumptions, deductions from assumptions, the gathering of data, inductions from data and so on. Especially delicate is the process of building probabilistic mathematical models. Here, 'learning by doing' is a good principle to follow. So it might be a good approach to require the students to make models for some concrete cases.

Teacher education should include project work. The best way to get the future teacher to understand the difficulties involved in undertaking projects is to include project work for students in the teacher-education programme. Here, again, 'learning by doing' is a good principle to follow.

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Jagat Narain Kapur

Mathematics in other subjects

Interdisciplinary co-operation

The aim of the 'Mathematics in other subjects', or the 'Mathematics across the curriculum' movement is to bring to light examples of the use of school mathematics in other school subjects so that mathematics teachers can use these examples to convince their students that mathematics is relevant to the study of all school subjects, and to convince the teachers of the other school subjects that they can usefully employ mathematics to give quantitative foundations to their subjects. A further purpose is to ensure the uninhibited use of arithmetic, algebra, geometry, statistics, etc. in the study of those topics in other subjects which can benefit by such use.

Essentially the object of the programme is to strengthen interdisciplinary co-operation, to highlight the role of mathematics in all walks of life, and to produce numerate citizens who will not be afraid to use figures and quantitative data when it is appropriate to do so. The goal is to restore mathematics to its core position in the curriculum.

Co-operation between mathematics and science teachers was the subject of an excellent conference (Steiner, 1979). However, co-operation between teachers of mathematics and of school subjects other than science is no less important. The mathematical formulation of problems can be usefully employed by teachers of history, geography, civics, economics, commerce and languages. At the same time mathematics teachers can enrich their teaching considerably by drawing upon illustrations from these fields.

One essential capacity to be developed in students is that of 'Mathematical modelling of real-life situations'. The increased emphasis on 'Mathematics in other subjects' should certainly help to develop this capacity.

Examples from physics

Most of our contemporary mathematics was generated in a physical context. It was created in the solving of problems of physics. Consequently, physics provides the greatest stock of examples to illustrate and to motivate mathematical concepts.

As a first example, consider the motion of a particle projected vertically upwards in a vacuum under the influence of gravity with initial velocity 'u'. Let 'v' be the velocity at time 't', and let 'x' be the distance travelled until this time. Then we have these formulae:

$$v = u - gt \qquad (1)$$

$$x = ut - \tfrac{1}{2} gt^2 \qquad (2)$$

$$/v/ = \sqrt{u^2 - 2gx} \qquad (3)$$

The graph of (1) is a straight line. The graphs of (2) and (3) are parabolas. The slope of (1) can be interpreted as a measure of acceleration. The slope of (2) can be interpreted as the velocity at time 't'. The area under (1) can be interpreted as the distance travelled. These examples can be used to motivate the concepts of derivatives and of definite integrals.

The velocities at the end of successive seconds are in arithmetic progression. The distances travelled in successive intervals of time are also in arithmetic progression. Again, if we know 'x' but wish to find 't', we solve (2) to get

$$t = \frac{u \pm \sqrt{u^2 - 2\,gx}}{8} \qquad (4)$$

If $x < u^2/2g$, there are two real positive roots of equation (4). So there are two times at which the given height 'x' will be attained, one in the course of the upward journey and the other in the course of the downward journey. If $x = u^2/2g$, the two roots coincide. In this case, the one value of 't' is taken to reach the highest point. But, if $x < u^2/2g$, both of the roots are complex. This shows that this value of 'x' is a height which can never be reached.

The introduction of the parabolic curves (2) and (3) provides an opportunity to introduce applications of mathematics to parabolic mirrors in telescopes, to the construction of solar heaters, and to other practical applications of the parabola.

This first example has illustrated the relationship between the physical concept of motion in a straight line and the mathematical concepts of the graphic representation of straight lines, parabolas, derivatives and integrals, together with arithmetic progression, quadratic equations, and real and complex numbers.

As a second example, consider the equation which links the values of the pressure, the volume and the temperature of a 'perfect' gas, namely,

$$PV = RT \qquad (5)$$

It involves three variables, the values of the pressure 'P', the volume 'V', and the absolute temperature 'T', together with one constant, the 'gas constant' 'R'. The opportunity can be taken here to introduce the concepts of variables and constants, and of functions of several variables. A teacher can also use this situation to motivate the concept of partial differentiation.

If the pressure (or the volume) is held constant, the graph which represents the values of the volume (or the pressure) and of the temperature is a straight line. But if the temperature is held constant, the graph of the values of the pressure and the volume is a rectangular hyperbola. However, the graph of the values of P and of 1/V is a straight line – and this is much easier to identify than a rectangular hyperbola.

The first graph can be used to introduce both the concept of an asymptote, and of transforming an equation of a curve to the simpler equation of a straight line. For example, the equation $y = ax/(x + a)$ becomes $v = u + a^{-1}$ if for 'y' we put '1/v' and for 'x', '1/u'.

As a third example, we consider the periodic phenomena in physics, like simple harmonic motion. They include the propagation of waves in water, of light, sound or electromagnetic waves, the production of sound as in making music, etc. All these make use of the simplest periodic functions, the sine and cosine functions. This shows that the applications of trigonometric functions in discussions of periodic phenomena is of greater importance than their applications to heights, distances and surveying.

Some other examples of the use of school mathematics in physics emerge from discussion of a more general equation of state, of Hooks' law, relativity theory and electrostatistics. These are discussed in Kapur (1977).

Examples from biology

The old fallacy – that there is no meaningful interaction between mathematics and biology – has been completely exploded now that agricultural, biological and medical scientists are employing mathematics, statistics and computers on a large scale and since mathematicians, statisticians and engineers need to possess a good knowledge of biology to be able to contribute to genetics, to population dynamics, biofluid dynamics, agricultural statistics, biomedical engineering, etc.

One of the most fascinating applications of mathematics is to human physiology itself. Given less than a dozen experimentally observed facts, we can, by using school arithmetic, algebra and geometry (Kapur, 1978*a*), answer questions like the following:

What is the volume of blood in the human body?
What is the total length, surface area and volume of all blood vessels?
(The total length exceeds that of the circumference of the earth at the equator, the total area exceeds that of a badminton court and the total volume is about 5 litres).
What is the total number of red corpuscules in the human body?
What is the total weight of red corpuscles formed in a lifetime?
How much oxygen do we breathe in one day, and how much energy does it provide us?
How much oxygen do we inhale, and how much carbon do we exhale in our lifetime?
How many air sacs are there in the lungs, and what is their total surface area and volume?
How does the diffusion of oxygen and of carbon-dioxide take place in the body?

We can formulate hundreds of similar, interesting problems and solve them by using arithmetic, algebra, geometry, statistics, mensuration. They are likely to prove more fascinating than the artificially-created problems from commercial arithmetic that we give the children today. All the problems suggested above concern the children's own bodies and so are of great interest to them. The human body is a wonderful organism and the problems it offers can excite the imagination of children and provide them with a strong motivation for learning mathematics.

When students know something about probability, the whole world of genetics lies open before them. They can understand how genetic characteristics are passed on, from one generation to succeeding generations, and how mathematical considerations can, through careful breeding, help to improve the species of plants and animals.

At a more elementary level, the study of patterns in plants can lead to the discovery of the wonderful Fibonacci sequence. This can, in turn, lead to more mathematics (Kapur, 1974).

The population and ecological models discussed in a later section also exemplify the interaction between mathematics and biology.

A comprehensive discussion of mathematical models in biology and medicine is given in Kapur (1974, 1978*b*, In press).

Examples from geography and astronomy

Interesting problems about times taken to travel by road, by train, or by air between different cities in a country can be constructed. By consulting bus, railway and airline time and fare tables, students can be asked to plan alternative tours, to find their costs and the times taken. Such projects provide interesting experience in using arithmetic, algebra and geometry, and will enhance their knowledge of geography.

Similar problems arise in international travel. But, here, other interesting problems arise because of the spherical surface of the earth, so that shortest distances are along great circle arcs. Human and commercial geography, and international trade provide a number of interesting mathematical problems at the school level.

Geographical map-making is essentially a mathematical problem, though it sometimes involves college-level mathematics. Secondary school students can be taken as far as possible in the principles of geographical map-making, and can be motivated to learn higher mathematics to achieve particular projections. We can have maps which show distances accurately, or angles accurately, or areas accurately. These may be called geometrical maps. We can also have population maps, or economic resources maps. In a population map, countries preserve their shapes as far as possible. Continguous countries remain continguous, but the sizes of countries are proportional to their populations. In area maps, the United States and the Soviet Union are bigger than India; but, in a population map, India will be almost three times their size. On the other hand, in a map showing per capita income, India will be among the smallest countries in the world. Such maps require a great deal of mathematical ingenuity to construct, but if each student were to construct two maps, so that a class of thirty students would prepare about sixty maps in a year, the educational value of the maps would be quite large. In fact, economic and social imbalances in the world can be brought out graphically and dramatically by such maps.

Astronomy fascinates school students, and many of its truths can be easily explained mathematically to students. Thus, knowing only a few observational facts, students can find out the masses and the distances of planets and of the moon, and discuss methods of finding the distances of stars. Knowing three formulae of spherical trigonometry, they can discuss the shape of the moon on different days, and they can prove that the length of the day is given by

$$\cos H = -\tan\phi \tan\delta, \qquad (6)$$

where ϕ is the latitude of the place, δ is the declination of the sun and the length of the day is 24 H/π hours, if H is measured in radians. This simple formula will enable students to discuss the variation of the length of the day throughout the year at different places of the earth.

Examples from history, civics, economics and commerce

Representing historical data of the regions of kings and of dynasties by bar charts and histograms can give useful insights to students. The representation of different areas controlled by a king or a dynasty at different times may also be instructive. The students may also try to represent quantitatively economic and social conditions in different periods of history.

In politics, the strength of various corporations, legislative assemblies, houses of parliament and of different combinations in voting and data on elections at various levels, etc. can provide many fascinating mathematical problems which may appear relevant to the politically conscious students of today.

The economic development of a country during the last two or three decades can only be highlighted through numerical data and its graphical representation. The tasks that have yet to be undertaken by a developing country can be understood quite well through numerical data.

Banking, insurance and taxation can provide a large number of examples. Problems can be constructed about optimizing the interest earned on a given amount of capital by a judicious choice of policies of investment, or about the purchase of commodities on a hire-purchase basis. There are different insurance policies available, and one can work out the differences between the benefits offered by the various policies. Persons have to choose between pension and provident fund schemes. And, in order to determine which scheme best suits one's personal circumstances, a great deal of calculation of compound interest and annuities is necessary.

Examples from language, arts and crafts and sports

Students can determine the relative frequencies of use on various pages of their textbooks of the various letters of the alphabet or of groups of these letters such as diphthongs, and discuss their stability. They can also compare mathematically the styles of various authors. This can provide a motivation for learning some probability theory, statistics, information theory, mathematical linguistics, codes and cyphers.

The knowledge of projective and transformation geometries can be of great value in the study of painting, sculpture and architecture. The seven groups of symmetry of the border designs, the seventeen groups of symmetry for planar design and the various groups of symmetries of polygon and polyhedra can give an additional dimension to the study of commercial art. Examples of Escher's art can be fascinating and rewarding from the points of view both of mathematics and of art.

Various technical crafts depend heavily on school mathematics, and interesting examples can be easily found.

Sports is an extra-curricular activity very dear to the hearts of students. Many interesting examples of the use of arithmetic, algebra, geometry, probability, statistics, combinatorics and graph theory can be provided by various sports.

Examples based on the use of difference equations

School students want to be convinced of the utility of the mathematics *they* study. They are not interested in the applicability of undergraduate or postgraduate mathematics. Most of the applications of eighteenth, nineteenth and early twentieth century mathematics depend on differential and integral equations, and, as such, cannot be discussed at the school level. Indeed, this fact has generated the belief that the applications of mathematics can be studied only at the university level, and that mathematics at the school level must be pure mathematics, or core mathematics, or even abstract mathematics. It was this fallacy which was responsible for the failure, to some extent, of these new mathematics.

However, some recent mathematical models in economics, history, political science, population dynamics, ecology, etc. depend on recurrence relations or on difference equations. These, with the help of school algebra, can be easily discussed at the secondary or the higher secondary school level. We illustrate this fact below with some examples:

(a) The births and deaths in a population in a year are proportional to the size of the population at the beginning of that year. So, the growth of the population in one year, which we can write as

$$x(t+1) - x(t) = bx(t) - dx(t).$$

$$\text{Hence, } x(t+1) = (1+a)\,x(t), \text{ where } a = (b-d). \qquad (7)$$

This generates a sequence:

$$x(1) = (1+a)x(o),\ x(2) = (1+a)^2\ x(o), \ldots, x(t) = (1+a)^t\ x(o). \qquad (8)$$

Thus, the population increases in a geometrical progression. This is the well-known law given by Malthus in 1798. However, this law cannot be indefinitely true. This is because, if it were true, there would be (at the present rate of growth) about 10^{15} persons in less than 800 years, and there would not be enough space on the earth for people even to stand shoulder to shoulder! However, the value of 'a' can, in the above equation (8), decrease with time, and can eventually approach zero. In that case, the population can stabilize at zero growth rate. The students can work out the various possibilities. Thus, for $x(o) = 4 \times 10^9$, and 'a' taking the successive values 0.020, 0.015, 0.010, 0.005, etc., they can find x(t) for successive values of t, such as 10, 20, 30, 40, 50. This can be easily done by using logarithms or pocket calculators. They can also work out x(t) for $x(o) = 4 \times 10^9$, a = 0.020 for t = 1 to 10, a = 0.018 for t = 11 to 20, ... a = 0.002 for t = 90 to 100 and a = 0.000 afterwards.

(b) A more realistic law of growth of population than the one at (a) above is the 'logistic law'. This leads to a modification of equation (7) above, namely to the equation:

$$x(t+1) = (1+a)\,x(t) - bx^2(t) \qquad (9)$$

Here, the second term on the right indicates the effect of 'overcrowding', of 'density-dependence', or competition for limited resources. This law gives an S shaped curve, and a stable population having a final value equal to 'a/b'.

(c) When there is a 'prey-predator' situation, that is to say, there are two species, one of which eats the other (e.g. cats and rats, or foxes and rabbits), we find that, in the absence of prey, the predator species decreases, while in the absence of predators, the prey species increases. The mutual contacts lead to the increase of the predator species and to the decrease of the prey species. Expressed mathematically, this gives two equations:

$$\begin{aligned} x(t+1) - x(t) &= ax(t) - bx(t)y(t) \\ y(t+1) - y(t) &= -cy(t) + dx(t)y(t) \end{aligned} \qquad (10)$$

If we know x(t) and y(t), (10) determines x(t + 1), y(t + 1). So, for given values of x(o) and y(o), we can find, step by step, the values of x(t), y(t) at any time 't'. Students can easily do some sample calculations and verify that the populations of the two species vary almost cyclically.

(d) Similarly, when two species are competing for the same resources, we get

$$\begin{aligned} x(t+1) - x(t) &= ax(t) - bx(t)y(t) \\ y(t+1) - y(t) &= +cy(t) + dx(t)y(t) \end{aligned} \qquad (11)$$

and students can draw the graphs of x(t) and y(t). It will be found out that, as 't' increases, in general, one of the species tends to die out and the other species increases.

(e) The second order recurrence relation:

$$x(t + 2) = x(t + 1) + x(t) \qquad (12)$$

gives the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, . . .
in which every term is the sum of the two preceding ones. The sequence leads to a large number of applications of mathematics (Kapur, 1974).

(f) Similar examples arise in compound interest, in the calculation of annuities, the spread of epidemics, the spread of rumours, in discrete-time economic models, political situations, etc. Literally hundreds of applications can be brought within the scope of school mathematics with the help of simple algebraic recurrence relations. Many such examples are discussed by Kapur and Khan (1981).

Some other areas of mathematics of possible interest in other subjects

Probability Theory is essential for developing probability models in the physical, biological, social and management sciences. Many school mathematics courses include discussion of the Binomial, Poisson and Normal distributions. Consequentially, the many applications of these to quality control, the theory of errors in physical measurements, etc. fall within the scope of school mathematics.

Statistics has also been introduced into school courses. So, the statistical representation of data, curve-fitting to experimental data and applications of elementary sampling theory can be discussed in all fields.

Graph Theory can be introduced in a few lectures. This at once opens up a number of applications to tournament games, family trees, hierarchical relationships, and formation of cliques, systems of one-way traffic on roads, etc. (Ore, 1963).

Combinatorial Mathematics is already present in the form of permutations and combinations. It has many fascinating applications at an elementary level (Kapur, 1970*a*).

Linear Programming has already been introduced in schools and has a large number of applications in business and economics.

Operations Research techniques, including those for inventory control, replacement problems, assignment problems and transportation problems have already been shown to have a large number of applications which can be taught at school level (Kapur, 1968).

Binary Arithmetic is useful for computer arithmetic, and for solving some interesting puzzles.

Mathematical modelling

To be able to apply school mathematics in other school subjects in a meaningful way, students and teachers have to develop an insight into mathematical modelling, and become familiar with a large number of mathematical models. In a recent Unesco-supported publication (Kapur, 1983 pp. v–viii), the following models at the school level were discussed:

Finding the height of a tower;
Finding the width of a river;
Digging a tunnel through a mountain;
Enclosing maximum rectangular area;
Finding closed curve with given perimeter and enclosing maximum area;
Finding radius of the earth;
Finding distance and radius of moon;
Finding distances and radii of sun and planets;
Finding the distance of a star;
Finding the shortest distance between two points on the surface of the earth.

Motion in a straight line with uniform acceleration;
Vertical motion under gravity;
Motion in a vertical plane under gravity;
Motion in a circle with uniform speed;
Finding mass of the earth;
Finding mass of the sun;
Comparing periodic times of planets;
Motion of artificial satellites;
Law of reflection of light;
Law of refraction of light.

Growth of a population;
Growth of a population, when resources are limited;
Influence of pollution on population growth;
Influence of age-structure on population growth;
Prey-predator and host-parasite models;
Competition models;
Epidemic models;
Alveolar sacs;
Pulmonary capillaries;
Exchange of O_2 and CO_2;

Energy requirements of the human body;
Transport of oxygen;
Mechanism of release of oxygen.

Planning regional and national trips;
Planning international trips;
Making physical maps;
Making maps to represent economic conditions;
Christaller's central place theory model;
Agricultural land use model;
Models for industrial locations;
Population density models;
Riley's law of retail trade;
Transportation models.

Mathematical models in commerce, banking and insurance;
Linear and non-linear models;
Models for payment of interest;
Models for payments through annuities;
Models for payments through deferred annuities;
Effective rate of interest on an insurance policy;
Calculating insurance premiums;
Choosing the best portfolio.

The assignment model;
The transportation model;
Sequencing models;
Replacement models;
Inventory models;
Dynamic programming model;
Linear programming model;
Mathematical programming models;
Queuing models.

Mathematical models involving combinatorics;
Partitions of a given number;
Linear graphs and friendship patterns;
Directed graphs;
Latin squares;
Cell growth and polyminoes;
Transformation on a finite set;
Symmetries in polygons;
Other symmetries.

Historical and bibliographic remarks

Though most of the mathematics created before the end of the nineteenth century was developed for the purpose of use, mathematics curricula were developed in almost complete isolation from the curricula in other school subjects. It is true that arithmetic was applied to commercial problems, that geometry was applied to mensuration problems, and that algebra had some artificial and some real applications based on quadratic equations, permutations and combinations and the binomial theorem. Also physics teachers made a significant use of mathematics, and chemistry teachers made some use. However, teachers of other school subjects did not regard mathematical methods as relevant to their teaching. The situation is now changing. This is because mathematics penetrated into the biological, social, business and economic sciences during the first half of the twentieth century, and has done so at an accelerated rate since. The consequences have now gradually begun to percolate down to the school level. Part of the reason for this penetration was the large-scale use of computers in all sciences, and, similarly, the reason for percolation into schools is widespread use of pocket calculators. Another reason for a growing awareness of the need for co-operation with other subjects arises from the realization by teachers of other subjects that mathematical concepts can help in imparting a deeper understanding of their subjects to their children. It is also being realized that, with the growing complexity of modern life, mere qualitative descriptions of concepts are not enough. For the sake of rigour, robustness and precision, mathematical concepts must be invoked as often as possible.

At Nottingham University in the United Kingdom, a mathematics curriculum project, 'Mathematics across the Curriculum', was undertaken for a number of years. Efforts were made by a team of investigators to find applications of school mathematics in other school subjects. These efforts resulted in a book edited by Ling (1978). In September 1978, the Institution for the Didactics of Mathematics (IDM) organized a conference at Bielefeld on co-operation between science teachers and mathematics teachers. The Conference was sponsored by Unesco, the Committee on the Teaching of Science (CTS) of the International Council of Scientific Unions (ICSU), the International Commission on Physics Education (ICPE) and the International Commission on Mathematical Instruction (ICMI). The proceedings, edited by H. G. Steiner (1979), contain valuable ideas on co-operation between science and mathematics teachers. The Conference, however, did not discuss co-operation with teachers of non-science subjects. This co-operation is equally important. The year 1979 also saw the publication of the Unesco *New Trends in Mathematics Teaching,* Vol. IV. This contains a chapter on 'the interaction between mathematics and other school subjects' by a panel headed by H. O. Pollak (1979). This chapter gives an extensive list of references.

A parallel movement was laying increasing emphasis on mathematical modelling. Here, mathematical modelling was to be of topics in other school subjects as well as of all walks of life. Burghes (1981*a*, *b*, 1982, 1983), James and McDonald (1981) and others have written on the subject, and the efforts of this group led to the first International Conference on the 'Teaching of Mathematical Modelling', held in July 1983, at Exeter University, United Kingdom.

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Part II

Support for teachers

Ken Lewis

Support for teachers: the mathematics department

In order to effect curriculum change, and for mathematics teaching to improve and to develop, support is needed from all possible sources. Help can often be found at regional or national level; in addition to its course objectives, aid at this level has the advantage of taking teachers from their immediate environment to new and different situations where they meet others for the informal exchange of ideas. Usually the mounting of local education authority courses is expensive. It can therefore be considered relatively infrequently. For the participants, too, these courses are time-consuming, and often entail being several days away from home. The content of such courses is sometimes rather remote from the particular and very real problems encountered in the day-to-day teaching of mathematics in a school. On the other hand, they can be stimulating, and they provide, incidentally, a small extra qualification, showing that the teacher regards his or her work highly enough to spend time, and sometimes money, on its development. Unfortunately, when a teacher returns from a course, provision is not always made for communicating the information gained to colleagues.

However, centralized inservice courses must continue, even though these tend to be isolated events which are unlikely to contribute significantly to the overall quality of mathematics teaching in an area except in the long term.

Improvement in the general standard of mathematics teaching in schools must come from within the schools themselves through the efforts and attitudes of the people in the mathematics departments. A teacher working in a well-organized, progressive and lively department which questions all it does, will become a better teacher. Help from within school is more likely to be 'real' help, directly related to actual teaching situations. Assistance provided from outside the school is obviously valuable, but it might be less immediately applicable and have a more limited effect.

Responsibility for school-based inservice training rests with the heads of departments or mathematics co-ordinators, who must provide the necessary leadership and example. In order to do this, they, in turn, need help and support both from their education authorities and from their head teachers. Courses in management, administration, leadership and mathematics are required both for newly-appointed departmental heads and for those already in post. Departmental heads need inservice training too.

The head of department should read widely to keep up to date with modern educational practice and with new developments in mathematics education. He or she should also have a sufficiently light teaching load for visits to be made to other classes in order to observe, to contribute and to learn. This is a most important aspect: the administration of a department can be done in out-of-school time, but working in a class with other teachers cannot. The head teacher should do everything possible to enable the head of mathematics to share classes regularly with colleagues, even if it means extra staffing. Such an arrangement can be quite flexible. The departmental head can take over a class and hence release a teacher to work with another colleague. Team-teaching and pairing can arise from this. Above all, the departmental head must ensure that the documentation and administration of his or her unit is sound. Not only is it essential for the mathematics syllabuses to be available and updated regularly, but there must also be detailed schemes of work for each course. Methods of assessment, testing and reporting should be clearly defined, a homework assignment system agreed by all and standards of marking of exercise books made explicit. A policy must be laid down to deal with problems relating to work and behaviour in mathematics lessons, and this needs to include realistic sanctions which can be enforced when necessary. A policy is also needed to deal with 'cover' for absent colleagues.

Some schools have found it possible to produce a handbook for the members of the mathematics department, containing details of these administrative matters, as well as such things as a list of resources – reference books, equipment, worksheets and the equipment that is available. The existence of such readily-available written information will add greatly to the confidence of the new, weak or unsure teacher. It is also of utmost importance that everybody in the department knows what is expected of them.

Regular weekly departmental meetings are probably the most important contribution of all towards inservice training. Whenever possible, these should be timetabled as an integral part of the school's programme. It is at these times that the members of the mathematics department can experience a feeling of unity and of team spirit. An agenda should always be produced, and, although departmental business will often feature prominently on the list, care must be taken to ensure that it is not always full of such matters. Minutes should be kept, and members of the department should be given an opportunity to take the chair and to conduct some of the meetings. An informal atmosphere should be aimed for, and staff should be encouraged to bring their coffee to the meeting. In addition to routine matters, some possible items for inclusion are:

The departmental head should report on anything he or she has read concerning mathematics and mathematics education since the last meeting. All other members should be encouraged to contribute in this way from time to time.

Particular attention should be drawn to any advertised courses for which members of the department might apply.

The departmental head should report on visits made to mathematics lessons, and should make special reference to good work and to new ideas seen. The teachers concerned should then give more detail about these items, or should be prepared to do so at the next meeting.

Comment should be made on good classroom layout, and on the display of students' work on the walls and notice boards. The meetings can be held in different classrooms so that these features can be seen by all.

If a member has been on a course or on a visit to another school, this should be described and discussed, and, at times, a whole meeting might be devoted to such a topic.

Any films, videos, computer programs or other new resource material which has come to hand since the last meeting should be shown, and their relevance and usefulness determined. If it is decided to adopt the material, then the departmental handbook should be amended accordingly.

The departmental head should keep a list of topics drawn from the schemes of work and should be prepared to present these in turn to meetings for consideration of teaching method; others in the department can be asked to present topics.

Thought-provoking papers should be written and discussed; some recent ones seen were 'Thoughts on Algebra', 'Study Techniques' and 'The Arithmetic of a Multi-Storey Car Park'.

Visits can be made to other secondary schools and to primary 'feeder' schools.

'Outsiders' can be invited to meetings. These can be the mathematics adviser, head teacher or deputy, or teachers from other departments, particularly from departments where there are cross-curriculum links, such as science or craft-design technology. If the school is fortunate enough to be near to a university or to a teacher-training college, then visiting speakers from these institutions can also be asked to contribute occasionally. It might also be possible to invite a local employer to a meeting. These people provide new faces, new voices and new points of view as well as their own expert knowledge.

If a school has a system whereby the senior students take their final examinations and then leave a few weeks before the end of term, an opportunity arises here for longer inservice training sessions. Two approaches have been noted.

A whole-day course can be devoted to improving or to rewriting a particular scheme of work. Although all members of the mathematics staff will not normally be free for the complete day, there should be enough people available in the school for 'cover' at this time of the year. A recent all-day programme was:

Aim – To Review the Course for Slow Learners in Years 4 and 5.

Session 1. (a) introduction and background;
(b) appropriate teaching methods;
(c) continuous assessment, objective testing and the construction of a profile.

Session 2. (a) the content – basic, useful, peripheral;
(b) a provisional syllabus;
(c) schemes of work and topics to explore.

Session 3. (a) resources already available;
(b) extra resources needed;
(c) costing.

Session 4. (a) classes to be involved initially and the amount of time required;
(b) staffing;
(c) allocation, and details of tasks to be completed before the next meeting.

The meeting was joined by the deputy headteacher and by the head of the 'Compensatory Education' department. Each session was led by a different member of staff, and documents were prepared before the meeting. All participants went away with one or more topics to work on. Everybody enjoyed the meeting, and found it very valuable, although all thought that rather too much had been included on the programme, particularly in Session 2.

Another use of end-of-term time is a series of lesson-length meetings. These can be held at times when the senior students who have departed would normally be having mathematics lessons. This is easier to arrange than the whole day course, and causes less disruption. A recent programme had on its list over three weeks:

A visit by the county mathematics adviser.
A visit and a talk by a university school of education lecturer.
A visit to a computer terminal.
Mathematics and Craft Design Technology.
The use of Logiblocks.
Flow Diagrams.
The teaching of directed numbers.

The arrival in the department of a new or a probationary teacher sets special responsibilities for the head of department. It often happens that such people finish their college or university courses before the end of the school year. When this occurs, they should be encouraged to join the school on a full-time basis. Some education authorities recognize the value of this, and are able to arrange payment during the time. The head of department should lay on an induction course for them, and it is best if this is in the form of a proper timetable. It might include:

Observation, and help with specific lessons chosen so as to give an overview of the full range of classes and courses.
Attendance at department meetings. The probationer should be encouraged to tell members about the teacher-training course he or she has just completed. This might well contain new ideas in which the others will be interested. It will also help the new teacher to realize that he or she has something to offer.
Meetings with the department head to discuss classes and teaching methods seen in the school.
Familarization with resources, both departmental and general school equipment. This should include instruction in the operation of apparatus in common use such as the overhead projector, the duplicator and the computer.
Attendance at school assemblies.
Attendance at any parents' evenings which happen to fall during this period.
Meetings with those responsible for the welfare of the students. This could include attendance at meetings of form teachers, year heads, house staff and the careers department.
Meetings with the head teacher at the beginning and at the end of the induction course.

During the first year of service, probationary teachers should not have a full teaching timetable. Both the school and the education authority should provide active inservice training. This should include regular attendance at the local mathematics centre, or teachers' centre. Here, they will meet other new teachers and be able to discuss mutual problems. They will become familiar with the organization of the system of education provided by the authority, meet its officers and advisers, particularly the mathematics adviser, and learn what resources are available centrally. Seminars should be arranged on educational topics to continue and reinforce the attitudes of thought and study brought with them from their training. Other members of the mathematics department should be kept informed about these meetings.

The needs of the teacher who has been in service for a number of years must not be overlooked. Renewal is needed for even the best of teachers, and all will need assistance when there is a change of syllabus or other new development.

Staff should be encouraged to join one of the national mathematics associations and to attend their local meetings. After such a meeting, it is not uncommon to find different teachers in their various classes all trying out the same new idea. In addition to local meetings, these associations publish regular journals as well as occasional booklets, all of which should be circulated amongst staff and discussed at meetings. Release for short courses should be made possible, but the most valuable concession is to be found in the sabbatical term. Any teacher who has served for five years should be invited to submit a programme of study, if offered leave for a term. This could include both academic study in mathematics and professional research into teaching methods. Attendance at a full-time educational institution should not be mandatory, and the teacher should be free to pursue his or her own interests by way of correspondence, the Open University or private reading. At the end of the sabbatical term, a report should be produced.

Another approach to renewal which has been tried successfully is to arrange teacher interchange with a neighbouring school. This needs careful organization and the teaching capabilities must be well matched. Both schools must be enthusiastic parties to the plan. Interchange can last from a week to a whole term.

All these and other items such as running a mathematics club, organizing a mathematics competition and arranging regular liaison with primary feeder schools go towards making a department which will help, support and develop all its member teachers. Working together in an attempt to create a new course, a new topic or even just a new worksheet can be of immense value. Participants in such working groups almost always benefit personally from their membership.

A simple set of criteria cannot be listed to help evaluate the success of the inservice activities of a mathematics department. This can only be judged subjectively from the attitudes of both students and staff towards the subject. Success is likely to be reflected in examination results. It is most obviously noticeable on the walls of the classrooms and in the team spirit of the department.

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Jânos Surânyi

Mathematics teachers associations

Why teacher associations?

The main purpose of teaching is to provide an education adapted, as far as possible, to the needs and demands of the pupils and to interest as many of them as possible. It requires from the teacher a steady improvement in his or her own culture, teaching methods and knowledge of children. This, in turn, calls for the regular updating of the content of the teaching and the regular exchange of experiences.

Teachers are under pressure from public opinion because everyone who brings up children feels an expert in school problems, and declares his or her criticisms frankly. Agreements usually remain tacit. Even educational authorities are sometimes inclined to imprudent criticism and to giving vent to personal bias about, for example, the apparent working hours of teachers. These are only a few of the challenges which teachers have to meet. The needs both for professional advancement and for mutual support call for joint action. Teachers have therefore established their own associations. How do they work? How might they work more effectively? In what follows, three examples will be discussed, and, drawn from these, certain conclusions will be offered about the role and the function of associations of mathematics teachers.

The Bolyai János matematikai tarsulat[1]

The Bolyai János Mathematical Society, named after one of the discoverers of hyperbolic geometry, is the Hungarian association of research workers and of teachers of all levels. The main concern of its Educational Section is to help the work of the teachers, and to stimulate and further pupils' interest in mathematics. To this end, the Society co-edits two periodicals in collaboration with the Ministry for Cultural

1. Its predecessor, the Eötvös Loránd Mathematical and Physical Society, bears the name of the distinguished physicist L. Eötvös, one of the founders of the Society. In existence since 1891, it bifurcated after World War II, into two societies.

Affairs. One, *A Matematika Tanitása* [Mathematics Teaching] published fortnightly, carries mathematical, pedagogical and other articles connected with mathematical instruction, for teachers of all levels. A column of mathematical problems for teachers is also offered. The other, *Középiskolai Matematikai Lapok* [Mathematical Journal for Secondary Schools], is published ten times a year for secondary school pupils (14 to 18 year-olds). Its main purpose is to set problems for solution and return. The text of the problems appears also in English and in Russian translation.

The official journal of the Society, *Matematikai Lapok/Mathematical Papers,* appears twice a year. It sometimes carries articles connected with mathematics teaching, but its main function is to report on all activities promoted by the Society.

Each year, the Educational Section of the Society organizes a five-day meeting, the *Rátz László Conference*, so-named in memory of László Rátz (1862–1930), a long-serving editor of the student journal mentioned above. As a teacher, he introduced the elements of calculus into his classes long before they became part of the curriculum. Apart from some plenary lectures, the Conference runs in sections according to different levels and types of schools. The participants listen to and discuss expositions on mathematical questions, on innovations in curricula, textbooks, etc. During the opening session, the Beke prizes are awarded for significant achievement in mathematical education and for the popularization of mathematics. These commemorate Manó Beke (1862–1946), professor of analysis at the University of Budapest, who worked hard to improve mathematics teaching in Hungary and to popularize mathematics and physics. In recent years, the Society has initiated a mathematical contest for teachers: they are given information on a particular theme and are required to set problems relating to it. The prizes for this contest are also awarded at the Conference. Between 400 and 600 teachers attend the Conference, which, each year, is held in a different town. Authorities usually assist participation.

During the school year, the local branches of the Society organize lectures for teachers (and also for pupils), partly with invited speakers.

The Society has arrangements with the mathematical societies of other countries to promote exchange visits. This enables teachers, among other members, to travel abroad to study mathematics education and to exchange experiences. The possibilities, of course, are dependent upon the teachers' knowledge of other languages.

Mathematics competitions in Hungary have a long tradition, and they play an important part in the mathematical erudition of its youth. The Society organizes, or co-operates in organizing most of the national competitions, which range from the elementary through to the academic and university levels (Freudenthal, 1969, pp. 99–100). Some local branches also organize competitions. In this context, it must be emphasized that competitions have their benefits, but also their dangers (Freudenthal, 1969, p. 99–100).

Occasionally, the Society submits proposals to the education authorities. In connection with one such proposal, the Ministry suggested and the Society agreed to start a teaching experiment at the secondary school level (the ninth to the twelfth grade), the purpose of which is to gather experiences upon which a revision of the curriculum may be made to become operative at the turn of the decade. This experiment is based on pupil-activity and on the discovery method of learning. It is currently in action.

The national council of teachers of mathematics

The National Council of Teachers of Mathematics (NCTM) of the United States is an example of a society of a large country. As such, its possibilities are considerable, and it is also expected to act on a very large scale. It was founded in Cleveland in 1910 by the collective action of 127 teachers, and, in ten years, its membership had grown to about 5000. Further growth followed the Second World War, mainly in the late 1950s, when a general surge of interest arose in the improvement of mathematics teaching. NCTM now counts about 4600 teachers as members, but the number of subscribers to its periodicals is actually much higher. These periodicals are: *The Mathematics Teacher*, whose publication NCTM took over shortly after its formation, and *The Arithmetic Teacher*, established in 1954 as a forum for the exchange of ideas and a source of techniques for teaching mathematics in kindergarten through grade eight. Both periodicals appear monthly, from September to May, and they provide information on all aspects of the broad spectrum of the pre-service and inservice of teachers. A more recent periodical of the Council is the *Journal for Research in Mathematics Education*. Its purpose is to ensure direct communication between the findings of research and the utilization of its results.

In 1954, the *Mathematics Student Journal* came into being. It proved to be a very popular quarterly. But unfortunately, it stopped publication in 1982 for financial reasons. In its place, a student journal is published in a *News Bulletin* and teachers are free to make copies according to the interest of the pupils.

The journals represent but a fraction of the publishing activity of NCTM. The Yearbooks are very popular. They treat different actual problems of mathematics teaching, of mathematics and of its history connected with instruction. Volumes twenty-three and twenty-four, entitled respectively *Insights into Modern Mathematics* (1957) *and The Growth of Mathematical Ideas: Grades K-12* (1959), did much to help the dissemination of the mathematical concepts underlying the reforms of mathematics teaching in the early sixties. In this way, NCTM played an essential role in the process of renewal. Moreover, with the support of the National Science Foundation, it organized regional meetings and published a report on them entitled *The Revolution in School Mathematics* (1961). It also helped with films and texts and the retraining of primary school inspectors, and in many other ways.

NCTM includes a large number of affiliated groups throughout the United States and Canada with different specifications and different ranges of activities. One representative of each group participates in the Delegate Assembly, which is convened each year in conjunction with the NCTM annual meeting. The chief function of the Assembly is to provide a forum for the interchange of ideas among the groups.

The Delegate Assembly is a highly respected and influential recommending agency; it makes resolutions concerning the inner life of the Council, its various organs, different arrangements bearing on mathematics teaching, instruction, the mobilization of public opinion and any matters pertinent to mathematics education. It submits its resolutions to the Council's elected body, the Board of Directors, for consideration. These, when approved, are usually forwarded to the competent committee, commission, or task force of the Council for further action, or to the Secretariat to prepare a proposal for the Board.

According to the competence and authority of the Council, its President has the opportunity from time to time, to contact the White House (and other official authorities of the State) to make known any changes in NCTM policy and consequent proposals on questions relating to mathematics teaching.

These examples, though only some of the activities of NCTM, serve to illustrate in how many different ways it helps the work of teachers of mathematics; how it shapes opinion, and seeks to generate an awareness of the importance of its work for social evolution; how it promotes the improvement of mathematical education and grapples with the obstacles, the gravest among them being the shortage of teachers and the waning interest in the vocation of mathematics teacher.

The shortage of mathematics teachers is an acute and dangerous symptom of educational malaise all over the world. But it is, perhaps, most acute in the developing countries. We accordingly turn to a developing country for a third example of what an association of teachers of mathematics can do for its members.

The mathematical association of Nigeria

The Mathematical Association of Nigeria was founded in Ibadan in 1962. At the inaugural meeting – attended by over fifty people –

Professor K. Collard (1963) devoted his presidential address entirely to problems of mathematical education. His dramatic sketch of the situation then prevailing was essentially as follows.

The supply of trained mathematicians in no way matches the growing awareness of the need for them, nor for the rising demand for competent technologists. Without changing this situation, technological and social progress will be brought virtually to a halt. It would be vain to hope for very much assistance from elsewhere. So contingency plans should be made and implemented forthwith. But, if plans are to be feasible, aims and possibilities require careful examination.

Rapid progress is out of the question in circumstances where a secondary school has to count itself lucky if it has one graduate in mathematics. Evidently, it is beyond the powers of one graduate to keep a constant eye on the mathematics of the school and, at the same time, keep abreast of modern developments. Much could be done, however, by central planning, and by using, on a much larger scale, the techniques of mass education that science has provided. Using the resources of the media, together with carefully planning work books and teachers' guides – Professor Collard believed – much could be done to overcome the shortage of teachers, while helping the teacher to keep abreast of modern developments. Such a scheme would neither be ideal nor easy. But, at least, it offers, as a first step, an escape from a pressing situation.

The Constitution of the Association (1963) formulates its objectives as:

(a) To encourage and stimulate mathematical thought and research.
(b) To assist those engaged in teaching mathematics in Nigeria to improve the teaching of mathematics and to provide contact between teachers and examining bodies.
(c) To provide a link between those concerned with mathematics in educational institutions and those in other fields.

The Association's journal *Abacus* appears twice a year. It has become one of the main instruments in furthering the aims of the Association. While maintaining the emphasis laid on stimulating interest in and improving the teaching of mathematics, the journal accepts papers on purely mathematical subjects, too.

The constitution provides for an annual conference, in connection with which the annual general meeting of the association is held, where all ordinary and honorary members attending each have one vote.

Conclusions: the main directions of possible activities of associations of mathematics teachers

It would seem that the main functions of associations of teachers of mathematics are:

- to aid teachers in their daily work;
- to co-operate with all concerned in the task of improving mathematics teaching, and especially in validating the conceptions held by teachers of their role in education;
- to promote public esteem for teachers of mathematics;
- to ensure that the general public is both aware of the main problems which beset mathematical education and is correctly informed about them.

The association can provide literature for the teacher on topics which are directly applicable in the classroom, and it can provide material to stimulate able pupils, informing them on progress in mathematics, methods, psychology, etc. To this end, the prime means is the publication of periodicals, supplemented by books and booklets. Other means include influencing and co-ordinating the activities of publishing houses in the needed directions.

A further very important and useful activity is the organization of refresher courses, conferences and similar programmes. These serve to improve the knowledge and erudition of teachers. They also allow the exchange of teaching experience and the discussion of various problems of mathematical education. They can lead to the formulation of recommendations for the consideration of the appropriate body of the association which can then forward it to the responsible authorities.

Radio, television, film and other tools of communication can also be usefully employed to inform teachers. And these can, in certain circumstances, be very important, but they are no substitute for personal contact such as the dialogue between an audience and the speaker and between teachers themselves.

To improve mathematical education is no easy task. While it is clearly in the interest of teachers to make mathematical instruction as efficient as possible, what is 'efficient' is not sharply defined. Many teachers (and inspectors too) are convinced of the efficiency of their own teaching style, and cannot contemplate any need to change it. So they are essentially against change. As for the association, it has to stand up for the views of its membership. The association has thus to fight simultaneously on two fronts. On the one hand, it has to convince the conservatives that requirements have changed radically, and that they call for methods radically different from the traditional ones. On the other hand, it has to elaborate – usually entrusting a commission with the task – its standpoint, and support it before competent bodies. It is essential to have the opportunity to become involved with potential reforms at the planning stage, and not merely in the final proposals, for these seldom respond to criticism. The participation of an association in curriculum reform holds out the hope that what is ultimately decided will have the virtue of feasibility.

Support for reform can be promoted by using the tools and the means already mentioned: suitable literature, courses, conferences, discussions, etc. But, as to the influence of reform on instruction, it is difficult to generalize since education systems in different parts of the world vary greatly. Some, for example, lay down the content of instruction for every class. In others, only examination requirements are laid down. In the latter case, the content of instruction is influenced to a considerable extent by the available textbooks. So, here, the association has the job of influencing the publishers.

In any case, however, the influence an association can exert depends strongly on its prestige, which in turn depends upon the soundness of its standpoint, upon the proportion of all mathematics teachers who are members, and upon the esteem which mathematics teachers enjoy in the society at large. At present, teachers are not as highly regarded as they used to be. They are not well-recompensed for what they do, and, in novels and plays, they are often depicted as figures of fun. Moreover, people often believe that teachers have not much to do apart from giving lessons. Their 'work' is only an imitation of work. This belief is sometimes reflected in official regulations, and even in official declarations. For example, in some countries, teachers are paid only for the months of the school year, not for the summer months.

Another example is the case of the *gläserne schule* (school under glass) in Lower-Saxony, where the Prime Minister expressed his desire that 'teachers have once to work 40 hours a week in the school in order that the real burdens of the teachers, and, through it, the need for teachers becomes clear'. At the time of this declaration, the 40-hour week was actually being tried in some schools of different type and level in Hamburg, Bremen and other cities. The example of Hamburg is referred to in *Erziehung und Wissenschaft* (Aktion, 1980). Here, after careful planning and preparation, twenty-three teachers started to work (during two weeks only) in the school from 7:45 a.m. to 5:00 p.m., with a break for lunch of 45 minutes. This meant a working week of 42.5 hours. Each day, the teachers completed a 'self-observation sheet' recording what they had accomplished and what they were unable to do. The sheets were summarized and posted on a notice board for the other teachers, pupils and parents to see. The teachers involved were horrified to discover how much overtime they normally have to work. On a 42.5 hour schedule, they could not prepare themselves for more than 54 per cent of the lessons during the first week and 52 per cent during the second. On no day was the unprepared total of lessons less than 40 per cent – and many other duties remain unfulfilled too.

In conclusion, it seems evident that teachers must strengthen their associations because only thus can they surmount the various problems which go far beyond the capacity of individuals, acting separately, to have any influence.

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Fahmy I. Mikhail and Fayez M. Mina

The in-service education of secondary school mathematics teachers

Introduction

The explosion of knowledge, the rapid developments in all aspects of life and the vast advancements in technology have made the in-service education of teachers essential. It is not, therefore, our intention to discuss in this article the importance of the in-service education of teachers, either in general or for teachers of secondary school mathematics in particular. Rather, we think it more appropriate to concentrate on the basic principles and the essential guidelines for some programmes of in-service education for teachers of secondary school mathematics, in a manner which, we hope, may be relevant, particularly for developing countries. Our views derive from practical experience acquired in Egypt and in some Arab States over several years.

We shall start by setting out a rationale of the study. This will be followed by some evaluative remarks on four different models of in-service education for teachers of secondary school mathematics to be found in the Arab States. We then give our opinion on those prospective trends in mathematics education which seem important, and we conclude by suggesting guidelines for framing policies of in-service education.

Rationale

We are going to discuss, in this section, the basic ideas and principles on which any programme for in-service teacher education should be based. The most important of these principles are the following:

The in-service education of teachers, in general, must be looked upon as a sub-system of the wider system of curriculum development in the subject under consideration. In other words, the in-service education of teachers in any subject must be planned as a link in a general plan for the development of the curriculum of that subject as a whole. It must evolve from, and should be involved in, a more general plan of curriculum development. Needless to say, the development of any particular subject, of mathematics education, for example, is, in turn, a sub-system of the development of the whole education system in a specific society, and will be affected by the different and more general factors and needs of the society.

Any programme for the in-service education of teachers will have its own components: its objectives, material content, teaching strategies, education media, methods of teaching and means of evaluation. In making an evaluation, teachers should be trained to take into account those aspects of learning that are not normally given sufficient emphasis (e.g. the emotional and psychomotor aspects) as well as testing the upper levels of cognition (e.g., understanding, analysis, application, problem solving, etc.). These different components must have definite organic relationships. They should develop in the light of the results of feedback obtained from both formative and summative evaluation. The efficiency of any programme of in-service teacher education depends on the consistency of its components and on their relevance to the requirements of wider systems of curriculum development.

Programmes of in-service teacher education should be professionalized. This means that almost all such programmes, whether academic or educational or cultural, should be oriented to serve the needs of teachers in their actual work. They should, in short, give on-the-job training. More specifically, the mathematics included in a programme of in-service education should help to strengthen the understanding of those topics which are actually included in existing or prospective curricula. Likewise, the methods of teaching employed in the programme should be a model of the recommended methods. A formidable lecture on evaluation, for example, will be useless unless it is followed by a direct practical application to actual items taught in the classroom. Any talk about the cultural values of teaching the history of mathematics will be in vain unless it is linked directly to the contents of the existing mathematics curriculum.

A successful programme of in-service teacher education should be oriented towards self-education and research activities, because, however successful, such a programme is inevitably limited in time and scope. This implies giving priority to identifying relevant sources of knowledge in the area of concern, developing the trainees' interest in and ability to use these sources, and practising some methodological aspects of educational research. In short, teachers should continue to grow on the job through self-education.

Trainees undergoing in-service education are intellectual adults, who have their own experiences, points of view and professional problems. Thus a successful programme of in-service education should be characterized by:

- its dealing with actual and prospective professional problems;

– its encouraging free discussion and the exchange of experiences and points of view in such a way that trainees gain understanding and appreciation of the subject;

– its trying to remove any undesirable attitudes towards the teaching profession that trainees may have, and creating acceptance of innovations and of new trends in mathematics education.

Mathematics, as an educational subject, is characterized by many specific features which greatly affect all aspects of mathematics education and call for careful consideration in programmes designed for teachers. The most important of these features, we believe, are these:

– The contents of mathematics syllabuses comprise, to a large extent, sequential topics. If a student misses one part of the syllabus, it will not be easy for him to follow the rest of it. This feature of mathematics should be borne in mind by curriculum planners and textbook authors.

– There is always a gap between the mathematics reflected in syllabuses, both at general and higher education levels, and mathematics as it is viewed and practised by philosophers and research workers in the subject. Many problems in mathematics education and teacher education are due to this diversity of viewpoint. Naturally, we do not advocate a complete unification of view as between school mathematics and all other aspects of the subject, but teachers and educators should, at least, be aware of this conceptual gap and should attempt to minimize it in an appropriate way.

– At different stages in their education, serious questions are often raised by students about the unity of mathematics, its importance for further studies in other disciplines, and its relevance to their future lives. The difficulty in answering such crucial questions is due mainly to the abstract nature of mathematics itself. The conventional sequence of school mathematics does not encourage appreciation of unity in the subject. Thus programmes of teacher education should deal intensively with these issues, and should seek to answer them incidentally by, for example, stressing topics that are used as useful tools in the various branches of mathematics, e.g. vectors, sets, functions, graphs, analysis, etc., by using, when appropriate, more than one approach in teaching a certain mathematical topic, e.g. algebraic, analytical, synthetic, graphical and vectorial approaches, and by re-building some parts of the mathematics curriculum so as to demonstrate its applications in other disciplines and in everyday life. In this respect, co-operation of teachers of other subjects is always helpful.

– The impact on mathematics education of new discoveries and trends in the theory of curriculum development and of educational psychology are limited and far behind what might be expected. This again may be due to the nature of mathematics, or to a failure within the pedagogic sciences to cope with mathematical thinking. However, we believe that a deeper review of the pedagogic works of such writers as Bruner, Bloom, Piaget, Gagné, Ausubel, etc., could shed additional light on mathematics education.

Models of in-service programmes for secondary school mathematics teachers in the Arab States

Owing to limitations of space and the need to adhere to the aims and scope of the present chapter, we do not intend to discuss in detail all in-service education programmes for secondary school mathematics teachers practised in the Arab States. We shall, however, select four different models of these programmes, two of them planned for specific purposes and the other two provided on a permanent basis. We shall describe the characteristic features of each programme and offer some evaluative remarks in the light of the ideas and principles expressed above. One of the two special models will be discussed in detail, as we believe that it is a particularly satisfactory programme.

Training programmes for modern mathematics

The most important of the in-service education programmes for secondary school mathematics teachers is the one associated with the introduction of curricula of 'modern mathematics' in all the Arab States. This was in response to the world-wide movement that prevailed during the sixties to modernize traditional curricula. To educationists, developers and university professors in most of the Arab States, it has cost a great deal in terms of effort and time as well as financially.

This programme came as a direct consequence of the Unesco Mathematics Project for the Arab States (Unesco, 1969), which was accepted and sponsored by the Arab League Educational, Cultural and Scientific Organization (ALECSO) (Jurdak and Jacobsen, 1981). Implementation led to national study groups and committees being set up to plan and organize in-service training programmes for the secondary school teachers responsible for the new curricula. These programmes were organized at different levels on a regional or a local basis, and were sponsored and administered by the ministries of education and the universities in the different states.

The characteristic features of this programme can be summed up as follows:

The programme is conducted either on the basis of part-time or of intensive study sessions.

The programme is carried out in several sessions, each of which is concerned with a particular text-book, e.g. the Training Session for Book 1, the Training Session for Book 3, etc.

The programme used to be, on the whole, restricted to the subject matter of the text-book under consideration, with, in a few cases, some more advanced relevant topics. The text-book used to be the main educational resource of the programme.

The programme adopts the didactic method of lectures being given by specialists in the subject.

More recently, a limited number of lectures has been added, devoted to the 'methods of teaching mathematics'. According to our own experience in Egypt, these lectures follow no definite syllabus and are left completely to the discretion of the lecturers themselves.

In few cases, the programme has one or two closing seminars which are used to discuss some relevant problems under the supervision of top administrators and authoritative personnel.

At the beginning of these programmes (in the early seventies), there were regular meetings between the teachers in charge of the new syllabuses and their tutors; these meetings were considered to be an essential element in the direct training course.

It is evident from the above brief outline that this in-service programme is an ad hoc model. It reflects few of the basic ideas and principles stated above. Nevertheless, it has had marked effects on the development of school curricula and has played a considerable role in the dissemination of new ideas and trends; we would not, therefore, wish to exclude such ad hoc programmes of in-service education. On the other hand, we would like to emphasize the need for such programmes to be re-built on a wider basis, to follow a clear line of thought and to use the method of workshops and seminars rather than of lecturers.

The SEC programme

This is a special programme organized by the Science Education Center, (SEC) Ain Shams University, Cairo. SEC's main function is the development of science education for all educational levels in Egypt, including mathematics and technology. One of the essential means to this end is 'to conduct in-service training programmes for science and mathematics teachers in order to develop a system that will lead to science teaching and curriculum development becoming a continuous, self-correcting procedure'. It started in the academic year 1974/1975, and continued for six years, with mathematics being introduced after three years.

Several secondary school teachers, senior teachers and supervisors (inspectors) (ten to fifteen in each discipline) are released by the Ministry of Education for one year on study leave to SEC. They are selected by a joint committee of SEC experts and Ministry authorities from hundreds of applicants from all over the country.

One of the characteristic ideas that distinguishes SEC's policy is the deeply-held belief that any kind of educational reform or development, in order to be successful, must be planned and carried out by integrated teams. These should include university staff who specialize in the subject matter, curriculum developers, psychologists, etc., as well as specialists from the National Center of Educational Research (NCER) and the Ministry of Education. The gains from this approach were clearly reflected in SEC's programme for the in-service education of secondary school teachers. The programme has two main aspects:

To improve the subject-matter competence of the participants:

Realizing that many trainees were university graduates of long standing – 20 years in many cases – it was deemed advisable to update their knowledge of science, both in general and in their own specific science disciplines. This was accomplished by organizing short courses of two kinds: general courses and specific courses.

The general courses were for all participants. They were concerned with topics of world-wide interest such as: energy, the universe, waves, elementary particles, ocean resources, micro-organisms, etc.; environmental pollution; principles of cibernetics; the concept of equilibrium and its applications in the different fields of science; the philosophy and characteristics of science; relationships between science and economic and social problems; relationships of science and technology. The scholars conducting these short courses were carefully selected from among distinguished Egyptian scientists.

The special courses, which related to the area of specialization of the individual participants, emphasized the recent developments and modern trends in the subject. During these courses, the trainees often paid field visits to various research and computing centres to observe applied research in progress. Research reviews were often prepared and presented by group members.

To improve the pedagogical competence of the participants:

Since it is intended that participants should take an active part in developing science and mathematics curricula in Egypt, it was deemed important that they should understand what is meant by 'science' and 'curriculum development'. To this end, classes were organized to deal with such topics as: curriculum design (defining the objectives of teaching mathematics in an operational way, criteria for selecting subject matter, designing teaching units, students' books, teachers' guides, etc.); means of evaluation; teaching strategies; visual aids and classroom activities.

One of the main successes of this programme, in our opinion and as we practised it, was that, in the process of building up the various units comprising three levels of secondary mathematics, the above ideas, principles and modern trends were all manifested in a practical way, through workshops, seminars and open discussions. In fact, the SEC programme has notably contributed to the preparation of senior educators in science and mathematics who are now leading various projects of reform in the Ministry of Education. Unfortunately, the programme, after continuing for three years, was stopped owing to changes in policy on SEC's activities.

Programmes for university graduates

Every year, particularly in Egypt, hundreds of newly graduated persons in pure science and in mathematics join the teaching profession directly. They are considered to be educationally unqualified. A special in-service programme was accordingly established to orientate the new teachers educationally and professionally. This programme is carried out on a local basis, and is administered by the different Governorates. As originally planned by the Ministry of Education in Cairo, it lasts usually for about three weeks and comprises seventy-two lectures covering a wide range of subject matter. Of these, eight lectures are devoted to 'methods of teaching'.

We doubt if such a 'mixture model' of training can achieve its goals, as no particular theme links its activities. The outcome of such a programme is – at the best – some educational and psychological knowledge, without any clear perceptions of its direct application to actual teaching situations. It may succeed in removing some of the barriers between young teachers and their pupils, and it may create some healthy attitudes towards the profession itself. Such programmes should not be excluded. Rather, they should be rebuilt. They should start with actual classroom situations and problems, and the programme should be designed in the form of workshops, seminars and open discussions, rather than given as lectures.

Internal mission programmes

Another ad hoc model of certain training programmes is that designed for senior mathematics teachers pending their promotion to the rank of supervisor (inspector) and for supervisors pending their promotion to senior supervisor. These programmes are provided, in Egypt, by the Faculty of Education, Ain Shams University, Cairo, for one year on a full-time basis. They are known as the 'Internal Mission'.

The programme aims mainly to upgrade the trainees in the subject matter, and it is usually devoted to the study of a particular mathematics textbook. Without wishing to be controversial, we believe that these internal mission programmes could contribute more than they do to the development of mathematics education in Egypt if radical changes were introduced to the content of them according to the principles stated above.

Important prospective trends in secondary school mathematics education

Before drawing conclusions from the above discussion and proposing a number of guidelines for designing profitable in-service education programmes, we think it would be helpful to set out what seem to be the most important of the prospective trends in secondary school mathematics education. These trends should be reflected in any in-service training programme, if the programme is to be in harmony with future needs. The major trends would seem to be these:

The re-building of mathematics curricula on the bases of the applications of mathematics in everyday life, its relationships with other disciplines and technology, and its relevance to the economic and social problems of the country. This trend is not in conflict with the treatment of mathematics as an abstract discipline.

The rejection of the argument about 'modern mathematics' versus 'traditional', as well as of any call for 'basics' versus 'applications' or 'processes'. School mathematics should be taught as an integrated whole of all its aspects, and with each 'branch' interacting with the others. In order to solve a particular problem, all possible channels must be explored, thereby demonstrating the unity of the subject.

The readiness to introduce the various concepts of mathematics, no matter how advanced they are, as early as the need arises, and allowing them to develop, in a spiral way, towards higher levels of conception.

Providing more time in secondary school courses for the study of probability, statistics, preliminary ideas of linear programming and computer science, as these are considered timely topics.

Paying more attention to the individual differences among the students, and catering for brilliant, retarded and handicapped as well as normal students.

Introducing radical changes in the methods and techniques of teaching mathematics, giving more consideration to multi-media systems of instruction and encouraging the self-education of students.

Some guidelines for policy formation

In the light of what we have said so far, we shall conclude this article by suggesting some guidelines, which, we hope, may be helpful to those with responsibility for policies of in-service education of secondary school mathematics teachers.

Goals and aims

Before launching any education programme, one must define carefully its aims, distinguishing between general aims and specific ones. The following are some of the important general goals of an in-service education programme:

To help to develop in teachers a capacity for continuous self-education, that is, to provide a means whereby teachers become 'life-long students of mathematics and pedagogy'.

To strengthen teachers' attitudes in favour of the teaching profession, education research and innovation.

To impart the capability to carry out evaluation processes and to make use of the results to improve teaching.

To impart the confidence and capacity which are necessary to enable teachers to design and to make their own educational materials and teaching units, and to motivate them towards activities which will enhance their teaching.

Elements of the programme

The programme may deal with some of the following major areas of concern:

The nature and philosophy of mathematics as a discipline and as an educational subject.

Mathematics education as a subsystem of other wider systems, e.g. the system of general education, planning of national needs, etc. In this, special reference should be made to the inter-relationships between the different systems.

Components of the secondary school mathematics curriculum with particular reference to contemporary trends in their teaching.

Various applications of secondary school mathematics in other disciplines and in technology.

The planning of mathematical units and lessons.

The building up of multi-media systems for teaching certain topics in secondary school mathematics curricula.

A means of understanding and appreciating the unity of mathematics as it is manifested in secondary school curricula.

Computation.
Methods of testing and evaluation.
Remedial teaching, and the teaching of special classes.

It is obvious that all these areas cannot be covered in one training session. The focus of interest of a training session will vary according to the specific purposes of the programme and the particular needs of the trainees.

Media

The main educational media recommended for an in-service education programme are: a few short lectures; seminars and discussions with specialists and popular figures in the relevant fields of study; workshops in small groups (the principal medium); individual assignments, e.g. writing short essays or research papers, producing educational aids, etc.; micro-teaching sessions; visits to computing centres, research centres, factories, museums, etc., in addition to any social activity.

Evaluation

During the programme, the administrating body should continuously carry out an evaluation of the trainees, taking into account the products of the different tasks they are given, their individual assignments and their contributions to discussions and to group activities. It is also helpful to have an attitude scale, to be administered pre and post-training, to test any changes of attitude. The whole programme, or any part of it, can be assessed by questionnaire so as to use the opinions of the trainees for making corrections along the way or to store for future reference. Needless to say, these suggested guidelines are intended to be flexible and capable of modification and adaptation to the particular needs of the trainees and to the facilities available.

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Part III

Two case studies

Gail Jaji, Levi Nyagura and Mike Robson

Zimbabwe: support for tutors, utilizing video and printed modules

Introduction

Teaching is still very much a matter of role-playing. Young teachers are strongly inclined to 'teach as they were taught', that is, to base their initial classroom performance upon dim memories of how their own school teachers once acted. To the extent that this is true, we have a vicious circle in which new teachers recreate and eventually hand on the old-fashioned teaching methods of the past. Furthermore, many teachers have a rather restricted vision of their role. They tend to see the teacher as a 'performer' – someone who acts in front of large groups of children. This chapter describes an attempt to provide student teachers with a far wider repertoire of roles to choose from through the use of inexpensive video and photocopying technology.

Rationale

We recently watched a teacher educator in action with a group of students, all training to be secondary school teachers. She said:

> Now what I am about to say is very important. It will almost certainly come up in the Examinations, so I suggest that you write it down.
>
> (The group took out their pens . . .).
>
> 'In the new, modern approach to teaching . . .'
>
> (They wrote it down as she spoke . . .).
>
> 'In the new, modern approach to teaching, we, as teachers, no longer dictate notes to children. Instead we arrange resources in such a way as to enable children to discover things for themselves . . .'

The reader might find the anecdote rather amusing, were it not for the fact that it happens to be true. So the question we must ask ourselves is this: what effect will that bit of training have on the students concerned? Will they do as the lecturer said? Will they build enriched learning environments for their pupils? Or will they do as the lecturer did, and dictate notes for examinations?

The incident, perhaps, illustrates one of the problems of teacher education. Teaching is, in part, a role-playing activity. When the new teacher, in his first school, gives his first lesson, what will he do? We suggest that he will play a role borrowed and derived from memories of teachers he has observed. But, if the only teachers he has ever seen are the small group of those primary school, secondary school and tertiary teachers who actually taught him (themselves probably teaching along lines laid down even earlier by their own teachers), he will presumably act very much as they did, and teach in the mode that was fashionable in the past. To break into this closed circle, it would seem that one of the more important tasks in teacher education must be to ensure that students observe a number of really excellent teachers at work in conventional classrooms. In this way, they can each select elements of teaching style from a variety of good models, and assemble new teaching roles of their own based upon examples of really good teaching. However, such exposure is not always easy to arrange. Excellent teachers are around, but they are a little thin on the ground.

In the sixties, teacher educators began to tackle this problem through the use of the new video-tape recorders, and by developing micro-teaching programmes. McKnight (1980) describes it thus:

> In a typical micro-teaching programme, after a technical skill of teaching has been described for the trainee through a videotape, the trainee's subsequent brief teaching performance is videotaped and then reviewed under various supervisory and other feedback contingencies . . . [later] the trainee teaches it again, usually to a different group of pupils.

Micro-teaching has now lost some of its glamour. Workers such as Galassi (1974) have suggested that video models, in addition to demonstrating desired behaviour, also present distracting cues. These compete for the students' attention and thus interfere with learning. It is as through video recording carries too much information.

But there is another problem. In our experience, most young teachers have a very restricted role in mind. They tend to imagine that teaching is that which they do when they act in front of a class. They don't see teaching as a process of organization, in which, through careful planning and skilled administration, the teacher devises and arranges learning situations for pupils. Instead, they see their role primarily as that of a performer. For example, students on teaching practice seldom set any homework. But, if they do, it tends to consist of a demand for pupils to record an image of the teacher's performance. 'And for HOMEWORK', they shout at the departing throng, 'For homework, complete your noooooooooootes.' An accomplished teacher, in contrast, plans in advance his homework tasks and exercises as the central focus of a carefully designed series of lessons.

If we take an orchestra as an analogy, then the teacher identifies with one of the performers, and micro-teaching is quite rightly concerned with developing the skills of the performer. But an accomplished teacher is more like the conductor of the orchestra, needing familiarity with a complex syllabus and scheme of examination, and having to plan a route through the syllabus best suited to the teacher's own abilities and those of the pupils. The teacher, too, must organize and control the work of assistants in such matters as ordering, receiving, storing, using and maintaining various pieces of equipment. Traditionally, all this is something that the teacher is supposed to learn on the job. But is it not possible that video-material, being very information-dense as suggested above, might be suitable as a medium for training young teachers in the role of 'orchestral conductor'?

One of the authors developed this idea for use in training science teachers (Robson et al., 1980). When it proved relatively successful (Robson, 1982), it was decided to explore the possibility of extending the technique to mathematics education.

In essence, the idea was to create for secondary school mathematics a teacher-training package based upon an actual lesson given by a good teacher in a local school. By prior arrangement, our camera team joined a teacher's lesson and made not only a complete visual and audio record of what happened in the lesson but also photo-copies of all relevant documents, both pupils' and teacher's, as found in the classroom. The collection of information was then condensed into a multi-media teacher-training module. This was published in a very limited edition for trial in a small number of teacher-education centres.

Description of the materials

The package (Jaji et al., 1982) consists of three major elements: a videotape, a booklet of collected documents and a set of eight question booklets for students. Each of these will be described in turn.

The videotape is in colour on low-band U-matic format (Pal system). It runs for 30 minutes and contains extracts from an unrehearsed, unscripted maths lesson.

The Document Booklet has thirty-four pages. It contains a copy of the actual lesson plan for the recorded lesson taken from the teacher's file, samples of two of the worksheets used in the lesson, three mathematics syllabuses currently in use in Zimbabwe, extracts from textbooks, the teacher's scheme and record of work, his test plan, the test associated with the lesson and copies of questions (cut out of past papers set by the external examination board) related to the concepts covered in the lesson.

Each of the Students' Question Booklets provides an 'input' and a 'demand'. The 'input' consists of comments and observations about the lesson element under study. The 'demand' consists of a series of questions to be answered after making a careful study of the relevant documents in the collection or in some cases after making a review of just part of the video tape.

The first booklet assumes that the student teachers will watch the whole videotape of the (condensed) lesson. Accordingly, it poses questions which focus mainly on the overall structure of the lesson, the classroom 'atmosphere' and the apparent attitudes towards mathematics and towards learning displayed by teacher and pupils.

Booklet 2 assumes that the student will review just the first part of the videotape. It, therefore, poses questions centred around the discussions taking place in the video lesson before the start of the practical work. Special areas examined include language, teaching aids, interactions between teacher and pupils and between pupil and pupil, and the concepts to be developed in the practical work which follows the discussion.

The third booklet focuses on the practical work. It assumes that the student will review that section of the videotape which records the practical work given during the lesson. It asks questions which require the students to analyse the Worksheets and assess their cognitive demands. Further questions focus upon the major purposes and features of what might be called the 'mathematics laboratory' mode of instruction.

Booklet 4 looks at how the teacher gets and uses information obtained by pupils from the practical activities. It invites students to devise further real-life examples which the teacher could have used to consolidate the concepts being developed.

Booklet 5 focuses on pupils' notes, and upon their homework. Students are encouraged to devise ways of improving the Worksheets so that these will lead pupils towards more effective note-making. They are also encouraged to study the relevant section of the pupils' textbook and to explore ways of integrating the use of the textbook into the lesson. The purpose of homework as a component of an instructional programme is discussed, and samples of pupils' actual written homework are studied and marked.

Booklet 6 concentrates upon an exploration of the teacher's syllabus, the scheme of work, the records and lesson plans, and Booklet 7 deals with assessment. It introduces students to the particular techniques needed to analyse the cognitive demands being made on pupils by the tests and the examination questions in the package. Finally, Booklet 8 involves the students in making an evaluation of the package itself as an element of teacher education. It invites them to propose useful modifications in the presentation and content of the package as a whole.

Using the package

The package was tried out with a group of twelve graduate students studying at the University for the Certificate in Education. To complete the work occupied five two-hour sessions, although students were required to carry out additional work on their own between some of the sessions. At the time of the trial, all of the students had already had at least one term of teaching experience, and some of them had had considerably more teaching experience than that. Each of the five sessions typically involved students for part of the time in the study of extracts from the videotape, and in the study of documents from the collection. This would be interspersed with a brief presentation by the lecturer, and by round-table discussion of the issues raised. The videotape generated a lot of interest, but it was not the most important part of the package. Far more time was spent on the study and discussion of the materials in the document collection.

It is impossible at this stage to offer a properly controlled and completely objective assessment of the value of this teacher training package, although it is hoped to do so eventually. However, a subjective assessment of the package may be of interest to the reader.

The main aim of the unit was to provide students with a holistic view of a teacher at work; to enable them to see a teacher not just as an accomplished performer in front of a group of children, but to provide a backstage-view of an adroit organizer of learning resources at work. In this, we believe that the package helped us to succeed where we had failed before. In writing afterwards about their experiences with the package, most students listed specific elements of the video-teacher's methods and techniques which they intended to adopt. Particularly popular ideas for adoption were the use of worksheets, improved forms of homework, the use of group-work, the use of a wide variety of teaching aids, project work and well-planned tests. Other ideas mentioned (but by fewer students) were those of planned work on the chalkboard, the ratio of teacher talk to pupil talk (our teacher spoke relatively little), methods of encouraging questions from pupils, moving round the class while the pupils are working, using a specification grid in order to assess the cognitive demands of test questions, and the making of detailed schemes and records.

In general, then, we think that the package has merit, not least because it provides concrete and readily understood examples of ideas which might otherwise be discussed in vague, abstract terms. We found, for example, that a discussion about meaningful and challenging homework can become quite lively when you have in front of you some specific questions for homework and some specific examples of children's responses to those questions.

What next? We should like to retain this package as the core of our maths education programme. But we intend to develop additional modules which will show a wide variety of maths teachers at work, and will illustrate a wide variety of approaches. These will include the use of the axiomatic method at sixth form level, the use of ability grouping, the teaching of certain special topics, the use of the historical approach in mathematics lessons, and so on. Meanwhile, our first module, described above, is now available to teachers' colleges, and the additional modules will likewise be released for use once they have been developed and tested. In this way, we hope to improve the training of mathematics teachers in Zimbabwe.

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A survey of secondary school mathematics teaching in China

Introduction

The purpose of this chapter is to describe recent reforms in the mathematical curriculum in China and to give an account of the teaching materials used in ordinary secondary schools. This follows a brief description of the primary and secondary school systems, together with some statistics taken from the 'Yearbook of the Encyclopedia of China'.[1]

The primary and secondary school system

In China, children begin school at 6 to 7 years of age. The primary stage is now five years in duration; this will be changed to six years eventually. The secondary stage is in two parts: junior middle and senior middle school. The first of these takes three years to complete, as does the second since 1982.

The ordinary secondary schools fall into two categories: those which belong to provinces, municipalities, or to autonomous regions, and those which belong to rural collective ownership. In addition to the secondary schools, there are also vocational middle schools, agricultural middle schools, technical schools, teachers' schools, schools for the blind, deaf and mute, and schools for minorities and others. The actual numbers of schools and students are shown in Table 1. They reflect information collected in 1981.

1. Beijing, China Encyclopedia Publishing House, 1982.

Table 1. Numbers of schools, students and teachers

School Category	No. of Schools	Students (unit: 10,000)	Teaching staffs (unit: 10,000)
Ordinary middle schools	106,178	4,859.56	374.54
senior	24,447	714.98	
junior	82,271	4,144.58	
Primary schools	894,074	14,332.83	616.46
Other middle schools	5,787	152.99	37.17

The evolution of the mathematics curriculum of the ordinary secondary school

In the years immediately following the establishment of the People's Republic of China, Chinese versions of U.S.S.R. textbooks were used because there was not enough time to compile our own mathematics textbooks for the schools. At that time, six school hours per week were allocated to mathematics in each year of secondary schooling.

It was soon found that the use of the translated materials in no way met teaching requirements, and in the late fifties, the compilation of mathematics textbooks, based on the country's needs, began. By 1963, textbooks for separate mathematical subjects, such as 'algebra' and 'plane geometry' for the junior middle years, and 'algebra', 'plane trigonometry', 'solid geometry' and 'analytic geometry' for the senior middle years, were published, and came to be used in ordinary secondary schools throughout the country.

A summary of the current mathematics curriculum

In 1978, it was stipulated that the teaching of each subject in middle schools should, as far as possible, be unified in order to meet the needs of the rapid development of science and technology, and the creation of a modern state. Accordingly, a document entitled 'A mathematics teaching programme for ten-year full-time[1] middle schools (the testing draft)' was formulated, and, based upon it, the work of compiling new textbooks for each grade in the secondary schools began. The new series was completed and published by 1982. Each book has been used experimentally at least once in each grade of all the ordinary middle schools, and the series is now the definitive one for the whole country.

The characteristics of the reform are that mathematics is no longer

1. The words 'ten year full-time' used in the prescriptive document refer to the five years of primary schooling as well as to the first five years of secondary schooling.

presented in the curriculum as separate, differentiated subjects. Rather, it is taught as a unified subject called 'mathematics'. Attention, moreover, is specially paid to the following issues.

First, the reinforcement of basic knowledge. By 'basic knowledge' is meant the knowledge students will need if they are to participate in modern production methods and the construction of national defence. It is also the pre-requisite for learning modern science and technology. It must be acceptable to the students. Examples include the basic operations of algebra and trigonometry, and the properties and the drawing of geometrical figures, together with the necessary inferential reasoning frequently used in modern scientific technology. Such skills foster the ability to think logically. In contrast, those traditional items which were over-elaborated and are of little use in modern methods of production, or which are irrelevant to future study, are now abridged. The purpose of this simplification and adjustment of the traditional content of middle school mathematics is to enable the students to devote their efforts to learning and basic knowledge well, and, at the same time, to learning more about modern mathematics.

Second, the enrichment of advanced scientific knowledge. To this end, elementary calculus has been added to enable students to grasp the concept of change and to provide a foundation for the further study of modern mathematics. Moreover, the replacement of traditional mathematical methods by advanced methods leads to an improvement in the students' ability to solve problems. Other examples are the inclusion of a basic knowledge of probability, statistics and the elements of computer science. The teaching is permeated, as appropriate, with the idea of sets and of one-to-one correspondence with a view to improving the student's ability to think in abstract terms.

Third, the reinforcement of basic training. This is mainly intended to foster the ability to operate correctly and rapidly in the context of numerical computation and the transformation of algebraic expressions. Logical thinking and spacial visualization are fostered to some extent, as is the ability to use mathematical knowledge to analyse and solve problems.

A number of drills, exercises and reviewing exercises are provided in the textbooks for all grades.

The current content of mathematics in secondary schools

In the prescriptive document mentioned above, the content of the course which covers the two stages in secondary schools is set out as follows:

Junior Middle First Grade:
1. Rational numbers;
2. The addition and subtraction of integral expressions;
3. Linear equations in one variable;
4. Linear inequalities in one variable;
5. Linear equations in two variables;
6. The multiplication and division of integral expressions;
7. Expressing a quantity in terms of its factors;
8. Fractions.

Junior Middle Second Grade:
1. Intersecting lines and parallel lines;
2. Triangles;
3. Quadrilaterals;
4. Evolution of numbers and quadratic radicals;
5. Quadratic equations in one variable;
6. Exponentials and common logarithms;
7. Similar figures.

Junior Middle Third Grade:
1. The co-ordinate system;
2. The solving of triangles;
3. Circles;
4. Functions and their graphs;
5. Equations of lines and circles;
6. Figures in perspective;
7. Preliminary statistics.

Senior Middle First Grade:
1. Power functions, exponential functions and logarithmic functions;
2. Trigonometrical functions;
3. Trigonometrical expressions for the sum or difference of two angles;
4. Inverse trigonometrical functions and simple trigonometrical equations;
5. Figures in space;
6. Conic sections;
7. Polar co-ordinates and parametric equations.

Senior Middle Second Grade:
1. Systems of linear equations;
2. The properties and proof of inequalities;
3. Complex numbers;
4. Permutations, combinations and the binomial theorem;
5. Probabilities;
6. A simple introduction to number scale and logic algebra;

7. Sequences of numbers and limits;
8. Derivatives and differentials;
9. The applications of derivatives and differentials;
10. Indefinite integrals;
11. The definite integral and its applications.

A new reform project of mathematics teaching

In April 1981, the Ministry of Education looked into the effects of 'A mathematics teaching programme for ten-year full-time middle schools (the testing draft)', and the use of textbooks based upon it. Consequently, two documents were published: 'A teaching plan (testing draft) for full-time six-year system middle key-schools', and 'A revised version of the testing draft of the teaching plan for full-time five-year system middle schools'. Furthermore, it was decided that the five-year middle schools system would be progressively changed into a six-year middle school system. During the change, there will be a common teaching plan for the junior middle stage, and the school hours per week for mathematics courses in the three junior middle years will be five, six and six respectively.

The content of secondary school mathematics should allow for flexibility. Student's aspirations and interests are different. Some will become engaged in physical labour; some will continue their studies after graduation. Moreover, the levels of teaching ability in the different regions and schools of the country vary, owing to the vast size of the country. Accordingly, three different kinds of materials are provided. These are designed to cater for the optimal courses, which, in the six-year middle school system, are offered after senior grade two. In one of these courses, a single optional subject is offered and given with four school hours per week in both the senior second and third grade. Students in this course study mathematics for five periods a week in each of the three senior years. The other courses arise from the separation of students into two sections: a social sciences section and a natural sciences section. Those in the social sciences section study mathematics for five periods per week in the first senior year and for three periods thereafter. Those in the natural sciences section also have five periods of mathematics in the first senior year, but six thereafter. The three types of courses account for the three different kinds of materials. The actual breakdown of the time allocated to mathematics in the junior and senior years of ordinary secondary schools is shown in Table 2.

Table 2. The numbers of teaching periods allocated to mathematics *in ordinary secondary schools*

school hours per week / grade / subject	Junior 1st	Junior 2nd	Junior 3rd	Senior 1st I	Senior 1st II	Senior 1st III	Senior 2nd I	Senior 2nd II	Senior 2nd III	Senior 3rd I	Senior 3rd II	Senior 3rd III
Algebra	5	3	3	3	3	3	3		3/4	2		2
Plane Geometry		3	3									
Solid Geometry				2	2	2						
Plane Analytic Geometry							2		3/2			
Calculus										3		4
Algebra and Geometry								3			3	

In this Table, the roman figures I, II and III refer to the three types of courses described above. It will be seen that students in the social sciences section take 'algebra and geometry' together as one mathematics subject in their second and third senior years. This is also the case with all students in the second junior grade. In all other cases, the mathematics subjects are taught separately. The 3/4 allocation to algebra and the 3/2 allocation to plane analytic geometry in the second senior year of the natural sciences section refer to the first and second semesters of the year respectively.

Experiments in the reform of mathematics teaching in secondary schools

Since the syllabuses for all the subjects taught in the ten-year system were promulgated by the Ministry of Education, the new teaching materials have been on trial in all primary and secondary schools since September 1978. Now, primary and secondary schools are undergoing a transition from the ten-year to a twelve-year system. Moreover, the needs of modernization imply greater demands on secondary school graduates from both the higher institutions and from vocations of all sorts. Consequently, experiments in the reform of teaching of all subjects in secondary schools continue. In this respect, the reform of mathematics teaching is accepted more readily owing to the significant role played by mathematics in natural and in social science research, and the widespread applications of mathematics in technology and in the economy generally.

In recent years, various experiments in secondary school mathematics teaching have been conducted in schools. For instance, experimental teaching materials for secondary school mathematics, written by the 'Writing Group of Experimental Teaching Materials', have been on trial in certain schools in more than ten provinces. This writing group comprised members of the Mathematical Research Institute of the Chinese Academy of Science of Beijing Normal University and of Beijing Teachers' College. They were supported by the Ministry of Education and enjoyed the co-operation of Professor Xiang Wuyi, a professor in the Department of Mathematics of Berkeley College, University of California. In another experiment, the teaching material is based on the 'Syllabus of the Mathematics Teaching Experiment in Primary and Secondary schools (first draft)', written by the 'Teaching Reform Experimental Group' of the Central Institute of Educational Science. It has been on trial in several provinces and municipalities. Other material on trial in more than ten provinces was based on an experiment in self-study and guidance, carried out by the Psychological Research Institute of the Chinese Academy of Science in association with teachers' colleges and educational bureaux in Guangzhou and Beijing. In addition, various other experiments involving content and methods of mathematics teaching are being carried out in different regions and schools.

As it is in secondary school that foundations are laid, all experimental work must not only consider the consequences of the modernization of science and technology, but also the acceptability of new material by students and its suitability for their intellectual development. The purpose of our experiments is to identify the teaching programme, the teaching materials and the teaching methods which will both be consistent with the prescribed syllabus and serve the needs of modernization. New programme materials and methods will be adapted to the characteristics of the country and international levels of mathematical competence. Owing to the vast population, the enormous size of the country, and the varying levels of development both of culture and of education, the different regions must evolve a number of different programmes and alternative teaching materials. This accounts for the variety of experiments on different syllabuses, teaching materials and teaching methods now being carried out in accordance with the principle: 'Let a hundred flowers blossom and a hundred schools of thought contend'.

The experimental teaching materials written by some teams, as for example, those of the team representing Beijing Normal University Beijing Teachers' College and the Mathematical Research Institute of the Chinese Academy of Science (with the co-operation of Professor Wuyi) can be divided three ways: into three different lines (algebra, geometry and function); into two circulations (the three junior secondary years, and the three senior secondary years); and into four 'junctures' or 'transitions'. One of these can be labelled 'general properties of the number system', which involves the transition from arithmetic to algebra. A second juncture, that of 'experimental geometry' involves the transition from the use of set language and simple logic to proof in geometry. A third juncture, namely, vector geometry to algebraic geometry, involves the transition from geometric algebra to quantitative analytic geometry. And the fourth juncture, that of 'limit and approximation', involves the transition from 'constant' mathematics to preliminary mathematical analysis, or 'variable' mathematics.

People who have experience with this experiment believe that such a system is beneficial in that it raises the level of students, helps them to grasp basic knowledge and methods, and fosters mathematical liability. Every effort is made both to ensure that the content is simple and practical, and that its explanation is comprehensible, complete and logical, as well as to modernize content so that students will be able to acquire the essence of the content and the methods of the three most important topics in elementary mathematics – number, space and function. Apart from selecting the essence of traditional materials, a certain amount of the new content of modern algebra is also added. Special attention is paid to the systematic nature of the content, to its exposition and to training in basic methods. Stress is laid on learning and understanding the fundamental content, so as to foster the use of general methods to solve problems. Particular attention is paid to the general properties and the general methods of the number system and expressions of it, to experimental and vector geometry, and to approximation.

This experiment seems particularly to suit middle 'key-schools', where conditions are generally better. Students in these schools seem to acquire deeper insights than do those in other schools, as well as an improved ability to think logically.

The experiment in the reform of mathematics teaching administered by the Central Institute of Educational Science emphasizes the training of ability. This, it is considered, is ignored in a teaching method which, characteristically, employs a sea of exercises as its main tactic for imparting of knowledge. Accordingly, the main aim of the project is to raise the level of mathematics teaching. As to content, calculus, probability, statistics and operational mathematics are added, where mathematical ideas, mathematical method and the perspectives of modern mathematics are stressed in order to simplify the traditional content.

The experiment in self-study and guidance administered by the Psychological Research Institute of the Chinese Academy of Science (with the co-operation of certain provinces and municipalities), pays particular attention to giving training in self-study when knowledge and skills are being imparted. Materials have been written to assist teachers to organize systems of self-study, and the guidance given to them covers the underlying psychological principles. Three principles govern the development in students of habits of self-study, of ability, and of their intelligence. These relate to the student's active involvement in the process of learning, the dynamic nature of the teaching material, and the importance of the teacher's role in elicitation and direction.

This particular experiment is being tried out in schools of various types, including secondary schools with better conditions and general and rural secondary schools. In general, good results have been reported.

These are still early days for all the experiments now on trial. With the junior middle stage, they have at most been carried out for only two or three years. These experiments will eventually be extended to all junior and senior middle grades. Limited though they are, results so far show that, as compared with classes in general, the experimental groups have made gains in several respects. These include increased learning, the acquisition of some modern mathematical ideas, and the development of students' ability to think abstractly, to operate successfully, to visualize and to study more independently. There are, of course, still problems to be solved. However, as the experiments are being conducted on a small scale, research and modification can continue in parallel with experimentation and some revision of the teaching materials has in fact been made in the light of experience.

The training of teachers

It is not sufficient to have new teaching materials. Considerable attention must also be paid to teacher training if the quality of teaching is to be improved. The training of mathematics teachers for secondary schools is of particular importance because laying the foundations of mathematics will have profound influence elsewhere – in for example, the study of other courses, the quality of institutions of higher learning, and the modernization of science and technology. But the present quality of the teaching in the secondary schools falls far short of current requirements. All teachers, therefore, regardless of age, are facing the same need to improve themselves professionally. To this end, they are using either teaching materials written for wide consumption, or various kinds of experimental materials designed for mathematics reform.

Various measures are now being taken to provide opportunities for mathematics teachers in secondary schools to continue their education. For instance, institutes of teachers' continuing education or education institutes provided by local education departments have been established in every province and municipality. There are now 31 education institutes at the provincial level, 247 education institutes and institutes of teacher training at the regional (or municipal level), more than 1,000 teacher training schools at the county level and many teacher training organizations throughout the country. A teacher training network has been formed all over the country. The various centres run a variety of training classes for teachers in-service, including mathematics teachers, who, in some cases, are released from their posts during the period of study. Some institutions of higher learning also run training classes for teachers in-service, and some mathematics teachers in secondary schools are recruited as students into these institutions. Among such are the Educational Institute in Beijing and the Beijing Teachers' College, which provide mathematics teachers with various kinds of training classes for professional improvement. (In this case, the teachers are released during study from their regular post for one to four years). Many activities, such as brief conferences, lectures on different topics and exchanges of experiences contribute to the improvement of teachers. Teachers are also helped to improve themselves through television programmes, broadcasts and all kinds of correspondence courses. Reference material on mathematics teaching is printed and distributed by the People's Education Publishing House (Beijing), by educational units in provinces and municipalities, and by teaching or research groups of mathematics departments in the teacher-training institutions and in organizations engaged in research into educational science. Such material provides the undergraduate students of teacher training colleges and teachers of mathematics in ordinary secondary schools with insights into teaching methods, while a great variety of activities organized by teaching or research groups for mathematics in secondary schools are also of benefit to the continuing education and improvement of mathematics teachers. Vacation time is frequently used to train the teachers who are taking part in experimental programmes.

In this context, mention should be made of a conference on 'experimental teaching materials of secondary school mathematics', organized by the Ministry of Education during the summer vacation in 1981. During this meeting, the writers of teaching materials explained their purpose and how it was hoped they would be used. This enabled the teachers engaged in the experiment not only to acquire a general idea of how the teaching materials were constructed, their characteristics and their innovative features, but also to study, in depth, the teaching materials of different subjects separately. A similar procedure was again adopted during the summer vacation of 1982.

The re-training of mathematics teachers in the secondary schools of China seeks not only to improve the level of mathematics among students and the teaching methods of teachers, but also to infuse them with broad educational perceptions. Here, it is necessary to make it clear that the aim of mathematics teaching in secondary schools is to serve the training of physical labourers. This involves their moral, intellectual and physical development, the cultivation of a socialist consciousness, imparting scientific technological knowledge and the importance of a healthy body. The teaching should not be directed towards the training of a few mathematicians, but should take into account the various needs of other specialists in institutions of higher learning. It should also consider the different requirements of the various vocations in society, such as engineering technology and economics. The needs of the majority of students must be considered. The scientific and cultural level of the whole nation must be raised. This can be achieved only if teachers become competent. If they do not, the so-called 'tactics of the sea of exercises' will be adopted. Students will continue to be asked to work many exercises, to attempt difficult exercises and to spend a lot of time on inappropriate work. This is not the way to promote the growth of qualified personnel.

Biographical notes

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His work as an educator has been widely recognized in international circles, and he has served as Vice-President both of the Inter-American Commission on

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He is a member of several scientific societies and the author of a dozen books as well as about a hundred publications bearing on pure and applied mathematics, education, science policy, and on the history and philosophy of science.

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After leaving college in 1956, KEN LEWIS taught at a comprehensive school in London, the United Kingdom, where he established one of the first school mathematics laboratories in the country. He then went on to lecture at a teacher training college in Plymouth, and later at the University of Botswana, Lesotho and Swaziland. On his return to the United Kingdom, he became Project Officer of the Continuing Mathematics Project, based in the University of Sussex. But he decided to finish his career as he started it – at a secondary school, and he is now head of mathematics at Grainville School, Jersey, Channel Islands. Ken Lewis has conducted short in-service training courses in Kenya, Mauritius, Lesotho, Nigeria, Swaziland, Botswana, South Africa and the United Kingdom. He was one of the early writers of the School Mathematics Project (SMP) and has written numerous other textbooks for both the United Kingdom and Africa.

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