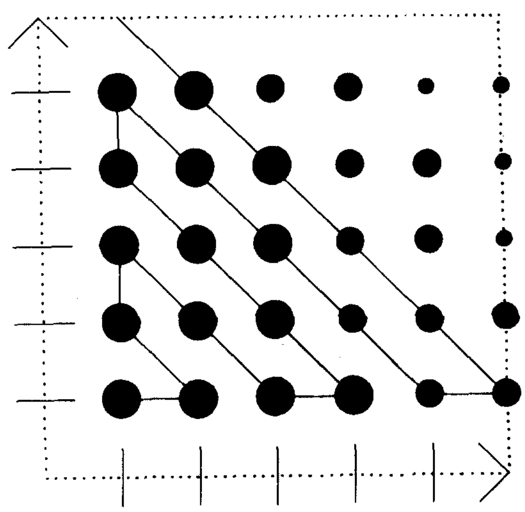
Volume 3

Studies in mathematics education

The mathematical education of primary-school teachers

Edited by Robert Morris

The teaching of basic sciences   Mathematics



Unesco

The teaching of basic sciences

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The mathematical education of primary-school teachers

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Preface

This third volume of *Studies in Mathematics Education* has been prepared as a part of Unesco's programme for improving mathematics instruction through the production of resource materials for those responsible for mathematics teaching. It is generally recognized that the teacher is the key to improving any mathematics teaching programme and, with this in mind, Volume 3 examines the responsibility of primary-school teachers for the mathematics component of the curriculum and the implications thereof for teacher education.

Volume 1 of *Studies in Mathematics Education,* published in 1980, described developments in mathematics as part of general education in schools of Hungary, Indonesia, Japan, the Philippines, the Union of Soviet Socialist Republics, the United Kingdom and the United Republic of Tanzania.

The second volume, published in 1981, examined the goals of teaching mathematics and considered how best to answer the question: does the teaching of mathematics correspond to the needs of the majority of pupils and their society?

In the present volume, Robert Morris, the editor, has unified contributions from sixteen countries while maintaining the individual style of each author.

Unesco wishes to express its appreciation to Robert Morris and to the many contributors to *Studies in Mathematics Education,* Volume 3. The views expressed are of course those of the authors and do not necessarily represent any position on the part of Unesco or of the editor.

Contents

Introduction

Teacher education is the weakest link in the chain of mathematics education. This was the unanimous conclusion of the Meeting on the Goals of Mathematics Education, a report of which formed part of Volume 2 of this series. It was accordingly decided to devote the whole of Volumes 3 and 4 of the studies to the theme of the education of teachers of mathematics.

In this volume, the main concern is for the needs of those who will teach mathematics in primary schools. In tackling this issue, an attempt is made first to expose the nature of primary-school mathematics. What are its special features? Whence does it draw inspiration? What winds of change are blowing upon it? What are the implications of the immaturity of the minds and the limited experience of the learners? When such considerations are brought out into the open, it is possible to perceive the tasks of the teacher and to make inferences about preparing teachers for them and about helping them to solve the problems they will encounter during their professional lives.

As with the earlier volumes in this series contributions were solicited far and wide. The views expressed have come from Africa, Asia, the Caribbean, Europe, Latin America, North America and the South Pacific. They represent the opinions of those with long experience, those with special experience and those with known particular interests. Collectively, the contributions combine the views of teachers, teacher educators, curriculum developers and research workers. They make a rich mixture; yet, surprisingly perhaps, they reflect a remarkable agreement about what primary mathematics should be. To this extent, it is to be hoped that teachers and others concerned with primary mathematics will feel sustained by the harmony of thought about the subject, and will find some of the procedures advocated applicable to and helpful in their own circumstances.

The first two chapters are general. The responsibilities of the teacher of primary mathematics are, it is suggested, to know the subject-matter thoroughly and to be aware of how it is used in daily life. At the same time, it is essential to understand the nature of childhood and how thinking evolves with growth. As for contemporary trends, these are seen as flowing from three sources: an increasing concern for the personal development of the young; the increasing availability of devices that remove the drudgery from computation; and the growth of knowledge about learning, about attitudes to mathematics, about the ways problems are solved and about the role of language in learning.

The third chapter discusses the importance of the environment as a source of problems that command attention and as a framework within which mathematics can flourish. In contrast, the fourth chapter is more concerned with what actually happens in the classroom: how to lead children along the road to discovery without disclosing anything about their destiny; the difference between informal learning and planned learning; the importance of problems as giving point to content as well as providing a challenge to thought; the place of problem-solving in the training programme—how it might be prompted and conducted.

The chapter on computers and calculators is likely to prove as controversial as it is stimulating. It sees the greatest potential of these aids as that of a tool for exploring mathematical ideas, capable of extending enormously the range and scope of mathematical activity in school. The brief history of aids to computation serves to place the contemporary versions in perspective while calling into question the need to teach the traditional algorithms of elementary mathematics. This raises two important issues for the curricula of the future: what the nature of 'basic numeracy' should be and how a 'sense of number' can be developed. Some suggestions are offered.

The four chapters that follow are concerned with concepts. The lead chapter attempts to clarify the meaning of the word and identifies three types of factors that make for conceptual difficulties: external factors, internal factors and linguistic factors. This is followed by a penetrating discussion of the difficulties that surround the concept of symmetry, an account of the difficulties that one group of children found with spatial concepts and a description of a method used to teach problem-solving in a situation where the language of learning was not a mother tongue.

The next five chapters are devoted to the pre-service and the in-service education of teachers. The first of the five tackles the issue of assessment, and finds a lacuna in what is known about the attributes of an effective teacher. This gap makes assessment difficult, to say the least. In lieu of knowing what has to be assessed, various criteria are suggested by which different teaching strategies might be clarified and gradually developed into a body of principles that would continue to evolve under exposure to increasingly refined levels of scrutiny. This chapter is followed by a case-study, that of a pre-service programme in southern Africa. In-service education is next discussed: its importance, its relationship to school work, and the implications for those who organize in-service courses and for those who run them. This chapter is followed by two case studies—one of an in-service programme in Africa, the other of an in-service programme in Latin America.

The volume closes with three chapters on support programmes for teachers of mathematics in primary schools. The first of the three discusses the role of teacher associations, with special reference to the Mathematics Association of Ghana; it also considers the usefulness of radio, with reference to a project in Nicaragua. The value of mathematics clubs in schools and colleges of education is then assessed. The closing chapter is devoted to the work of the *Instituts de recherche sur l'enseignment des mathematiques* (IREMs) in France. This falls into three categories: the retraining of classroom teachers, the production of pedagogical materials and undertaking fundamental research into the didactics of mathematics teaching.

Josefina C. Fonacier

The responsibility of primary-school teachers for the mathematics component of the curriculum: implications for teacher education

Man by his very nature is continually studying his environment: how to control it for the betterment of his life and how to preserve it for the sake of future life. And to communicate his ideas, language must be used. At the most rudimentary stage, descriptive words such as 'small', 'smaller', 'hot', 'hotter', 'above', 'a little higher', 'to the right', 'a little more to the right' are useful. When more precise descriptions are needed of the degree of smallness or hotness, or of how much higher, or of how much more to the right, then one enters the realm of mathematics. Mathematics makes possible the study and organization of observations, and it becomes a determiner for decisions.

So mathematics emerges from man's environment and from his experiences, and it may be applied to further study of the environment.

As man's intellectual power develops, he becomes capable of taking his emerging mathematics beyond his physical experiences to a more lofty level of thought involving concepts and symbolism and into a world of mathematics where relationships and principles are discovered through the manipulation of symbols. Much of the new knowledge he acquires is referred back to the practical level, to be used anew in his study of the physical world. Much, however, remains in the realm of mathematics itself, temporarily filed on the shelf of abstractions to be taken down later to develop more mathematics or if someone finds a use for it in the practical world.

The above paragraphs could provide a scenario for the mathematical component of the school curriculum. A problem in the physical world establishes the need to study some mathematics. The needed concepts and skills are then developed, reinforced or broadened, and fervent mathematical activity toward this development, reinforcement or broadening is undertaken. Finally, the mathematics that has emerged is used to solve the problem that provoked the search and to solve related problems as well.

The problems that need to be solved may differ from country to country or from age to age. Old procedures may be discarded as new or improved procedures are designed. So details of content beyond the basics may vary. But, no matter in what part of the world or at what point in time, the intellectural skills that underly mathematical activity are useful, even in non-mathematical settings.

The place of mathematics in the primary-school curriculum

At the very least, therefore, mathematics in the primary schools should, first, supply man with the basic mathematical content and skills he will need to tackle real-life problems. It should, second, cultivate thinking and reasoning skills, and so strengthen the intellectual underpinnings of human social interactions.

The cliché that was common among mathematics educators in the United States in the later 1970s was 'back to basics'. A position statement, made by the National Council of Supervisors of Mathematics (1978, p. 148), put the issue this way:

> The present technological society requires daily use of such skills as estimating, problem-solving, interpreting data, organizing date, measuring, predicting and applying mathematics to everyday situations. The changing needs of society, the explosion of the amount of quantitative data demand a redefining of the priorities for basic mathematical skills.

This statement clearly differentiates 'basic skills' from 'productive skills' or from 'job preparation skills'.

We shall not compile a list of basic concepts and skills. However, to mention a few, ratio and proportion should certainly continue to be on the list. There are signs of an increasing emphasis on decimals, and of a decreasing emphasis on fractions. And, with the universal adoption of the metric system, other systems are declining in importance, and, with this, converting from one system to another is falling out of fashion. The use of tables and of graphs is becoming more popular with the media. The enlightened person must be able to read and interpret them, so it is important to include among the basic skills the reading and interpretation of tables and graphs.

The proposition that 'estimating' should be included among the basic skills can be culled from the computational errors exemplified below.

$$\begin{array}{r} 34 \\ -\ 19 \\ \hline =\ 25 \end{array} \qquad\qquad \begin{array}{r} 34 \\ -\ 9 \\ \hline =\ 35 \end{array}$$

Both examples illustrate a common error in subtraction, that of subtracting the smaller digit from the larger in the same column, regardless of the order. The child who is habituated to estimate will, at least, recognise that the second answer is faulty. The difference cannot be greater than the minuend. The presence of error in the first answer is less obvious, but it may still be detected by the more alert child who is trained to look critically at the reasonableness of his results.

That problem-solving should be among the basic skills is recognized by even traditional teachers. Look at these problems:

(a) $50 \div 12$.

(b) A group of scouts on a camping trip brought 50 kilograms of rice. If this rice should last 12 meals and the same amount is cooked each meal, how many kilograms should be cooked each meal?

(c) A group of 50 scouts went on a hiking trip. They came to a river that could be crossed by raft. If the raft could carry 12 people at a time and was loaded as fully as possible on each trip, how many scouts rode on the last trip?

In these problems, the computation to be performed is the same in each, but the skills they invoke are different. Problem (a) calls only for numerical drill. The situation described in (b) is clear and straightforward. It requires comprehension of the written words, and then the skill needed for (a). Problem (c) is less straightforward than (b). Comprehension is involved, but so also is more thinking. It is in problem-solving that understanding and thought (as opposed to mechanical skill) are crucial.

The ability to think is basic to any human endeavour. In mathematics, we can invoke thought when developing perceptions of space, in measuring, in organizing data, in making and testing conjectures or through inspired guessing, and, of course, in making logical inferences. All these activities might be called the processes of mathematics because they are used as one develops mathematics. However, they are not unique to mathematical processes. They are used also in other subject areas and in other human transactions.

Besides the two broad goals of mathematics in schools stated above, there are goals related to the affective domain. There seems to be a growing emphasis on humanism—on the inculcation of values and attitudes. This trend can be fostered in mathematics. If so, the subject would contribute towards the overall aim of education: the total development of the individual as a person and of the individual as a member of society.

Josefina C. Fonacier

The responsibility of mathematics teachers for the mathematics component of the curriculum

The teacher's best teaching aid is herself. The teacher owes it to her pupils, and ultimately to society, to exploit this aid to the fullest possible extent. A necessary condition to teach mathematics is to know mathematics. First and foremost then, the teacher must have some competency in and an understanding of the basic content and the associated skills. She must also know what it means to do mathematics.

Educators consider mathematics as one of the best media for the development of thinking skills. The often-heard opinion that a person who is good in mathematics could be a good chess player does not refer to computational ability but rather to reasoning ability. The teacher must be able to give substance to this viewpoint (of mathematics as an effective medium for developing thinking skills) rather than just paying lip service to it. In the process of teaching mathematics, the teacher must be able to seize opportunities for developing the skills of reasoning, for developing in the children habits of organized thinking.

Mathematics should not be regarded as an isolated body of knowledge. The teacher should be able to see mathematics in the environment and in other disciplines. This is pedagogically important, since the teacher must use or provide experiences or situations that are the starting points for children to discover and develop the inherent mathematics. In technical language, we say that a physical situation is used to introduce and develop a 'mathematical model' of the reality of the situation.

On the other hand, the teacher who can readily see mathematics in the environment may be able to point to those different situations to which similar mathematical descriptions apply. With these, one mathematical model serves to give expression to several situations, just as the two different scouting situations described above invoked the same computation of $50 \div 12$. The teacher should help the children discover mathematics in their experiences, and should then help them to extend this mathematics to other situations that may be beyond their experience.

An often-quoted argument for the study of mathematics is that it is an important tool, much used in science and technology. It is not enough to tell this to the children. They must be convinced of it by exposure to specific examples. The teacher who is knowledgeable about the community and about the environment may be able to provide meaningful linkages between mathematics on the one hand and science and technology on the other, and therefore contribute to this conviction.

National drives and politics impact upon curricular programmes in mathematics. The teacher should become aware of national goals and policies and should then seek to bring about a closer relation between what is taught in the classroom and what is learned and done outside of it in terms of these national efforts, at least within the child's surroundings or community.

Examples might include development of the countryside, better nutrition, improved health and sanitation, and population education. Economics oriented activities may be simulated in the classroom. As Broomes (1981, p. 55) puts it:

> Not only do we need to devise methods which would allow the emergence of creative behaviours among teachers and pupils in the mathematics classroom, but, most important, the methods penetrate other subjects of the school programme and the external world. Moreover, the external world should impinge on the mathematics to be studied in schools.

Any situation in which teaching and learning take place should prompt the teacher to consider the interplay between the learner, the subject-matter, and the context or the community and society in general. We have yet to consider the learner. The primary-school teacher must be sensitive to the child. She must be aware of how the child understands certain concepts. She must be aware of the viewpoint of the child, the narrowness of his perceptions, his degree of abstraction, the language he uses. In short, the teacher must know the child.

Special teaching plans are needed to help a child understand 'volume', not only as something that depends upon the button of a radio set, which is within the child's experience, but also as a mathematical concept of the quantity of space occupied. Again, younger children are usually conditioned to think of multiplication as repeated addition. They, therefore, may find it confusing to conceptualize $\frac{3}{4} \times \frac{1}{4}$, where repeated addition does not apply. The sensitive teacher who anticipates this difficulty would plan her teaching strategy to broaden the idea of multiplication before confusion arises. A similar situation arises when children are drilled in subtraction to use the following language:

10 take away 1 is 9.
10 take away 2 is 8.
10 take away 3 is 7.

This drill emphasizes the 'take away' idea of subtraction to the exclusion of comparison (i.e., how much more, or how much less), which also invokes subtraction.

To cite one more example, it is not uncommon for a child who has difficulty in doing the division $600 \div 25$ to have little or no trouble in answering the question: How many 25-cent coins make 6 dollars? The teacher who is aware of this could capitalize on the money problem to teach the more difficult computation.

A child's perceptions are the starting points of his learning. The ideas the child brings into the classroom, as well as his background of experiences, affect how he will perceive new information. It is therefore important that teachers should elicit what their pupils already know about the concepts or the principles that are to be introduced. This knowledge of the pupils' perceptions will provide hints about where and how to begin teaching new ideas. It may also serve to link classroom teaching with the realities of community life.

Implications for teacher education

The above discussions point to the following concerns as ones it is particularly important to include in the training of mathematics teachers:

Competency in the concepts and skills that are needed in everyday living, in one's place of work and in understanding the community and the immediate environment (i.e. mathematics as knowledge).

Competency in the intellectual skills and in those habits of organized thinking which are needed when doing mathematics and are characteristic of human behaviour (i.e. mathematics as process).

The ability to see mathematics in physical situations, in other disciplines and in the environment (i.e. mathematics as a model or as a tool).

Awareness of those habits, values and attitudes that can/should be fostered in primary education, particularly in the mathematics component of it.

Sensitivity to the nature of the child—his background, his needs, his perceptions, his learning patterns, his capabilities.

Familiarity with the various teaching techniques that can be used to attain specific objectives—a repertoire of pedagogical skills that the teacher can draw on to cope with varying conditions.

Some competency in designing teaching techniques and strategies—teaching creativity and innovativeness.

Traditionally the type of thinking that is involved in doing mathematics was believed to be no more than deductive reasoning. Today, however, it is realized that other intellectual skills are involved in mathematical thinking, and that these are general to rational intellectual behaviour. Evidence of this awareness will be found in the many articles written of recent years on the inquiry approach. The development of skills of any kind, whether concrete or abstract, necessitates practice. So, to develop any of these skills, the teaching style should be one where the students can practise these skills. In the usual training course for teachers, the mathematics professor presents a well-organized, finished and nearly flawless logical body of knowledge. Little experience is given, therefore, of actually developing these intellectual skills. Invariably, then, the prospective teacher learns mathematics by one method, but is told to teach it by another.

The ideal mathematics teacher of prospective teachers is one who, in the process of his own teaching, teaches methodology by example. He should be an exemplar not only in conveying mathematical content but also in importing teaching skills.

In a recent survey of primary mathematics teaching in Philippine schools, it was found that the teachers' own proficiency in computation correlated with the level of the grade being taught. That is, teachers of the higher grades are more adept in computation than are teachers of the lower grades. The temptation is to jump to the conclusion that teachers of the lower grades are less well prepared in mathematics than are teachers of the higher grades. But there could be another explanation.

Computation is listed among the basic skills. The drills involved must be explained from several points of view. Not only must the mechanical manipulations be explained, but the reasons underlying the manipulations must also be explained. So, too, should the relations among the different manipulative techniques be explained. This multiplicity of explanation is especially necessary in the calculator-conscious society we have become today. The finding that teachers in the lower grades are less proficient at mechanical computations than are those in the upper grades could be related to disuse rather than to incompetency. So, whereas basic skill in computation has traditionally been synonymous with mechanical manipulation, a contemporary view would be that it is necessary to ensure that teachers in training are conversant with the necessary manipulations, with the reasons for them and with the relationships between them. Furthermore, it is necessary to ensure that teachers in training understand that it will be their duty to transmit all these understandings to their pupils when they become teachers themselves.

One other finding of the Philippine survey was the preferences teachers have for the various subjects of the school curriculum. Generally (and especially in the lower three or four grades), a primary teacher has to teach all the subjects of the curriculum, not only mathematics. One of the questions in the survey asked the respondents to name their first and second preferences of all the subjects they taught. Surprisingly, only a minority gave mathematics as their first or second choice. Teachers, it seems, teach mathematics because they have to. If they had a free choice, they would prefer not to teach it. This finding has implications for teacher education. If a training course could be designed that would inculcate (at least to some extent) in teachers an enthusiasm for mathematics, the issues of what to teach and how to teach might be more easily tackled. Attitudes can be contagious. A teacher with a liking for mathematics will unconsciously convey this liking in his or her teaching style, and infect the children with a similar liking. Liking what one does in school greatly facilitates the learning process.

Because a primary-school teacher is expected to teach all the subjects of the curriculum, it is inevitable that pre-service education is an amalgam of an extensive range of courses. But when the student goes into teaching, her subject preferences and teaching needs emerge. After some years of professional experience, the teacher begins to ask questions about her own teaching–hopes and frustrations, strengths and gaps. Furthermore, after some years of teaching, those who have inclinations for teaching mathematics begin to identify themselves. Encouraging and helping the mathematically inclined teachers would not only hasten their own individual growth; it could also contribute toward assisting other teachers. The opportunity to observe an exemplar of good mathematics teaching handling children may have a greater impact on another teacher's methodology than any quantity of psychological principles and learning theories gleaned from a book.

Clearly then a sustained process of in-service training is necessary. It is needed to assist all mathematics teachers in resolving their teaching problems. It is also needed to enable the more mathematically inclined teachers to become better exemplars, able to assist their other colleagues 'not by deluging them with instructions and advice, but by involving them in collective thought and constructive criticism of the teaching function, giving it a sense of responsibility and the means of fulfilling it' (Revuz, 1981, p. 99).

In retrospect

This chapter has attempted a *tour d'horizon* as an introduction to what follows. It has raised some of the topics and issues of the mathematics curriculum, of mathematics teaching and of mathematics teacher training at the primary-school level. These, and others, will be picked up and discussed in greater depth in the chapters that follow. It makes no claim to novelty. The topics it has touched on have all been raised before. If, however, a difference can be discerned between the present and the past, it is the new, conscious and organized effort of present-day mathematicians, mathematics educators and teachers to tackle what have been for too long matters of little concern to society.

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Hilary Shuard

Contemporary trends in primary-school mathematics: implications for teacher education

Goals for primary mathematics

Among the goals of education there are always some that emphasize the needs of society for literate and numerate citizens and workers, able to cope with the demands of technology in agriculture or in industry. Other goals emphasize the personal development of the individual as an independent person, whose education is planned to contribute to his total understanding and appreciation of life. In mathematical education, progress towards both types of goals begins in the primary school, and, for many children in many countries, all the formal mathematics they will ever learn is learned in the primary school. In the United Kingdom, in the Mathematical Association's (1956) far-sighted and forward-looking report *The Teaching of Mathematics in Primary Schools,* the goals were based firmly on the development of the individual as a thinker:

> The aim of primary teaching . . . is the laying of this foundation of mathematical thinking about the numerical and spatial aspects of the objects and activities which children of this age encounter.

This point of view was reinforced (Department of Education and Science, 1979) by H.M. Inspectors of Schools (HMI), in a discussion document on primary-school mathematics:

> We teach mathematics in order to help people to understand things better–perhaps to understand the jobs on which they might later be employed, or to understand the creative achievements of the human mind or the behaviour of the natural world. It is the particular power of mathematics that its central ideas help us to do all of these things.

This tendency to emphasize the development of the individual is apparent not only in the curricular goals of the developed countries. A worldwide view was stated in the report, *Mathematics Education at Pre-school and Primary Level,* given at the third International

Congress on Mathematical Education (ICME) (Athen and Kunle, 1977), in 1976:

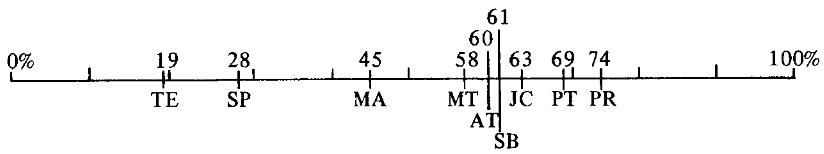
> Based on the fundamental idea that there is no difference between the nature of a child's thinking and that of a mathematician, a tendency is slowly, but surely establishing itself; this tendency is to replace the learning of mechanisms and their applications to standard problems by activities in which the child demonstrates research and inventiveness . . . appealing to children's wanting to understand, letting them develop their own research strategy and thus, experience the pleasure of solving a problem, mobilizing their knowledge and previous competencies and inviting them to propose new questions.

It cannot be denied that there are great difficulties in the way of implementing goals such as these, even in the developed countries. Surveying the state of primary-school education in England (Department of Education and Science, 1978) HMI found that, in about a third of the classes at all ages, children were spending too much time on the repetitive practice of processes they had already mastered.

The section of the 1975 report of the National Advisory Committee on Mathematical Education (NACOME) (United States) which considered computational skills stated:

> One of the strongest undercurrents in present curriculum decision-making is pressure to re-emphasise computational skill in arithmetic and algebra. The trend is undoubtedly a response to public criticism of declining computation test scores. Traditional school instruction far over-emphasised the facts and skills and far too frequently tried to teach them by methods stressing rote memory and drill. These methods contribute nothing to a confused child's understanding, retention, or ability to apply specific mathematical knowledge. . . . Many teachers of the 'new math' era sought improved skill performance through deeper student understanding of the structures underlying computational methods. . . . We have reason to suspect that in many classes teachers very poorly related structural understanding to algorithms embodying the structures. . . . The members of NACOME view with dismay the great portion of children's school lives spent in pursuing a working facility in the fundamental arithmetic operations.

This dismay is evidently not shared by the general public, the pupils' parents, or even by some of the pupils themselves. All these groups take the view that, if, by a particular age, children have not yet learned computational skills by drill and practice, further drill and practice are called for. For example, in the United States, the project of the National Council of Teachers of Mathematics (1981). Priorities in School Mathematics (PRISM), surveying opinion about priorities in school mathematics for the 1980s, found that members of the public who were concerned with education supported devoting increased time to drill and practice (see Figure 1), in spite of the fact that research evidence suggests that this is likely to be detrimental to achievement. In this survey, it should be noted that teacher educators and supervisors of mathematics in schools, two groups closely concerned professionally with mathematics teaching in the elementary school, were much less in favour of drill and practice than other groups. The group of professional mathematicians were the next of those who least favoured increased drill.



Key: PR principals of elementary and secondary schools
(part) PT presidents of parent-teacher organizations
SB presidents of school boards
MA college teachers of mathematics
SP supervisors of mathematics
TE mathematics teacher educators

FIG. 1. Percentages supporting increased time on drill and practice across content strands

The situation is very similar in many developing countries, where educational practice has traditionally emphasized rote-learning methods. Two quotations from Volume 2 of the Unesco *Studies in Mathematics Education* (1981) will indicate the world-wide nature of this problem:

> Although the the revision and reorganisation of the subject matter has potentially made mathematics more meaningful, the methods of lecturing and of rote learning are still dominant. The utilization in the elementary school of 'child-centred activity learning' and of manipulative materials is the exception rather than the rule. . . . National examinations play a central role in the future of the students. . . . The examinations in mathematics . . . samples questions which call mostly for lower cognitive processes. Thus the examination itself promotes memorization and rote learning. [Arab States] (Unesco, 1981, p. 144).

> Teaching methods are formal, emphasizing drill and the memorization of information; and the curriculum removes the children from their social and cultural milieu, emphasizing book learning and examinations. [West Indies, but written more generally about developing countries.] (Unesco, 1981, p. 47).

If there is no consensus between mathematics educators and the public (which includes the children's parents) about the goals of mathematics education, a serious problem is posed for teacher education in mathematics. Consider, for example, the dilemma faced by a young elementary-school teacher in the United States. Her pre-service training has convinced her of the importance of laying a foundation of mathematical thinking through practical activity. And she is committed to letting children develop their own research strategies. Yet she will find it very difficult to put her commitment into practice in the face of an insistence on drill and practice by the principal of her school, by the school board which chooses the textbooks and by the testing system of her state, which emphasizes goals she does not share. The same conflict of goals occurs in very many countries, developed and developing, around the world. Teachers absorb the climate of educational thinking that surrounds them in the places where they live and work. In 1977, Ashton repeated a survey (Simon and Willcocks, 1981) which she had first carried out in 1969, of the aims of primary teachers in England. Teachers were asked to place in order of importance a list of possible aims for primary education. The three aims in the list that related to mathematics changed their rated importance for the teachers over the eight-year period in the way shown in Table 1.

Table 1. Aims of primary education, rank order given by teachers

	1969	1977
Maths for everyday life	15	5
Computation	20	2
'Modern' maths	34	10

Mathematics, in general, became more important among the teachers' goals over this period. But computation, in particular, rose dramatically in importance, taking second place in 1977. It is unlikely that, during this eight-year period, the emphasis in either initial or in-service teacher training in the United Kingdom had radically changed so as to reflect a new emphasis on computation, especially by the drill and practice methods to which many teachers reverted during these years.

Technology and the primary mathematics curriculum

The easy availability of inexpensive hand-held electronic calculators has changed the arithmetical skills an adult needs to use in his everyday life. Calculations that cannot be done mentally are now almost always done by adults in developed countries with the aid of a calculator. By the time they reach the age of 11 or 12, large numbers of children in the United Kingdom and the United States, and in many other developed countries, either own a calculator themselves or have access to one owned by another member of their family. Small but increasing numbers of children now live in a family in which a microcomputer is one of the family's resources. The advent of this personal technology has had comparatively little impact on the primary mathematics curriculum in developed countries. In many primary schools in the United Kingdom, calculators are banned from the mathematics lesson, or, at most, are used to check answers obtained by other methods. No published primary mathematics scheme in the United Kingdom yet makes use of the potential of calculators for helping children to understand mathematics better. In the United States, the National Council of Teachers of Mathematics (1980) recommended in *An Agenda for Action: Recommendations for School Mathematics of the 1980s* that:

> Mathematics programs must take full advantage of the power of calculators and computers at all grade levels. . . . Calculators and computers should be used in imaginative ways for exploring, discovering, and developing mathematical concepts and not merely for checking computational values or for drill and practice. . . . Schools should insist that materials truly take full advantage of the immense and vastly diverse potential of the new media.

However, the same report prefixed its proposals on calculators and computers with the statement:

> It is recognized that a significant portion of instruction in the early grades must be devoted to the direct acquisition of number concepts and skills without the use of calculators.

It is possible that this rather cautious preamble has inhibited the development of curricula in which calculators are used to help children to understand mathematics. But it reflects the likelihood that a curriculum in which the calculator was used as a tool for developing mathematical thinking from an early stage would be misunderstood by the general public and by primary teachers, and would be unacceptable to them. It is still widely believed, by many primary teachers as well as by parents and the public, that the aim of primary mathematics is to enable children to perform the four rules of arithmetic competently by pencil and paper methods. Whatever the cause, many primary teachers in the developed countries see the use of calculators in primary mathematics teaching as a threat.

The survey of (Suydam, 1981) of the use of calculators in pre-college education (United States) points out:

> The biggest disappointment to many people is the scarcity of published materials in which the use of calculators is integrated throughout the curriculum. Instead, most materials suggest supplementary modules. . . . In few instances have calculators affected methodology—how mathematics is taught remains the same whether

> calculators are used or not . . . Instances of content change—where content has been added or dropped because calculators are available—are similarly slow in occuring.

In addition to the need to integrate the use of a calculator into the primary mathematics curriculum, the need to teach children to make effective use of calculators should also produce changes in the curriculum,. It has been found that many adults in the United Kingdom are alarmed by the display the calculator produces when a calculation such as 10 ÷ 3 is entered. So, in fear, they abandon the one tool that could help them to be more effectively numerate. The interpretation and use of the calculator's decimal display should form part of the teaching of primary-school mathematics. So should the skills of estimating the magnitude of an expected answer and checking the answer given by the calculator in another way. Both are needed for effective calculator use. A new emphasis on the ways of solving the mathematical problems of everyday life, and on using the calculator as a tool for doing so, will need to replace the old emphasis on the mechanics of computation. It is doubtful if the skills of written computation with numbers of more than two digits will now be needed. But the need for confident mental calculation will become greater.

In the developing countries, calculators are much less readily available, but at the third International Congress on Mathematical Education (Athen and Kunle, 1977) d'Ambrosio (Brazil) said:

> The use of calculators and computers in schools has to be considered a must and this not only for the developed countries, but for the developing countries . . . because school education there has to make up for a lack of experiences which is due to the economic situation of these countries. With respect to the influence of calculators on teaching mathematics, the importance of numeracy as a purely mechanical work has to be called in question. The chance to have more time available for a creative mathematical education in very elementary grades must be taken.

It is likely that teachers in developing countries will have attitudes towards the encroachment of the calculator on the traditional primary-school mathematics curriculum similar to those displayed by their counterparts in the developed countries. If calculators are to be used effectively in the primary grades, or if pupils who do not yet have them available are to become effective adult users of calculators, initial teacher education throughout the world will have to strive to ensure that student teachers develop positive attitudes both to the use of calculators in school and to the consequent major curriculum change in primary-school mathematics. Moreover, this curriculum change will have to be prefaced by massive in-service education, designed to help teachers to understand the new needs and to take a positive attitude towards them.

There is a danger that the practical and cheap personal tool of the calculator, which people will be able to use regularly throughout their lives as an aid to effective numeracy, will, with curriculum development, be overshadowed by the more spectacular microcomputer. As yet, few primary-school classes regularly use a microcomputer as a tool for mathematics learning. But development work is proceeding. Here too, primary-school teachers need to become flexible enough in their thinking to accept the help of microcomputers in their mathematics teaching, as this technology becomes available. Their acceptance should not be uncritical, for many of the early programs were designed to reinforce, by drill and practice, the former pencil-and-paper drills of primary-school mathematics, which are fast becoming outdated by the advent of calculators.

Another tool of the new technology, which many primary-school children in the developed countries already own, is a digital watch. The digital readout invokes new skills of interpretation different from those needed for an analogue watch. So learning to 'tell the time' will also need to change its emphasis. Here, too, the curriculum in many schools has been slow to react, and teachers are not always ready to look for advantages as well as disadvantages for the development of the mathematical understanding that may come from children's familiarity with the new way of displaying the clock.

The learning of mathematics

The new emphasis, in the last few years, on testing and evaluation has revealed that, in many developed countries, a sizeable proportion of primary-school children do not learn much of the material the mathematics curriculum expects them to learn. If the curriculum emphasizes an understanding of mathematical concepts, understanding is not always the end product. If the curriculum emphasizes the acquisition of the skills of pencil-and-paper computation by methods based on drill and practice, children often develop their own erroneous methods, which are certainly not those intended by the teaching. For example, in computation with fractions, large numbers of children around the world give answers that show that they apply mistaken rules in inappropriate situations. The report of the Assessment of Performance Unit (1981) (England, Wales and Northern Ireland) described the levels of response, in its large-scale survey of eleven-year-olds, to question on computation with fractions. Some examples are shown in Table 2.

Table 2. Performance on addition of fractions among 11-year-old British children

Question	% Correct	Sum of nums. / Larger denom.	Sum of nums. / Sum of denoms.
$\frac{1}{4} + \frac{1}{2}$	58	$(=\frac{2}{4})$ 4%	$(=\frac{2}{6})$ 13%
$\frac{1}{6} + \frac{2}{3}$	37	$(=\frac{3}{6})$ 6%	$(=\frac{3}{9})$ 21%
$\frac{2}{5} + \frac{3}{5}$	60	(both these methods give the same answer)	$(=\frac{5}{10})$ 13%

The style of teaching employed in many British schools is intended to emphasize understanding. So most children have some experience of cutting out and colouring fractions of shapes when they carry out activities designed to convey an understanding of equivalent fractions and of operations on fractions. Even so, the results of pencil-and-paper testing indicate that an appreciable proportion of children are not able, at the age of 11, to apply these ideas to abstract computations with fractional numbers.

The United Kingdom is not alone in possessing a problem with children's ability to compute with fractions. The National Assessment of Educational Progress (United States, 1979c) found broadly similar results among American 13-year-olds. This is shown in Table 3.

Table 3. Performance on addition of fractions among 13-year-old American children

Question:	% Correct	Sum of nums. / Sum of denoms.
$\frac{4}{12} + \frac{3}{12}$	74	(Not given)
$\frac{3}{4} + \frac{1}{2}$	35	$(=\frac{4}{6})$ 34%
$\frac{1}{2} + \frac{1}{3}$	33	(Not given)

In Scotland, the opportunity was taken, in 1978, to re-administer to 15-year-old pupils in the Central Region an arithmetic test which they had first taken at the age of 11. Among the 15-year-olds, 53 per cent could give the correct answer to

$$\frac{2}{3} + \frac{5}{6}$$

compared to the 38 per cent who had previously been able to do so at the age of 11 (Scottish Council for Research in Education, 1978).

In England (Hart, 1981), the CSMS project examined the ability of children between 12 and 15 years old to compute with fractions. It also tested their ability to solve problems in which the same computations were embedded in real situations. For example, in the same test as the computation

$$\frac{3}{8} + \frac{2}{8}$$

was given, the following problem was set: In a baker's shop $\frac{3}{8}$ of the flour is used for bread and $\frac{2}{8}$ of the flour is used for cakes. What fraction of the flour has been used?

It was found that the ability to solve addition and subtraction computations declines as the child gets older. The ability to solve the problems does not decrease with age and one is left with the hypothesis that the problems are solved without recourse to the computational algorithms. Many children do not, in fact, seem to connect the algorithms with the problem-solving and use their own methods.

The increasing volume of research like that cited above calls into question the suitability for inclusion in the mathematics curriculum for all children of primary age the skills needed to handle such abstractions as equivalent fractions. There certainly seems to be massive misunderstanding of these ideas among very many children. Shuard (1982) has carried out a similar comparison of the results of large-scale testing, in developed English-speaking countries, of the concept of place-value, a topic that is central to a child's understanding of number. She found very similar levels of misunderstanding. For example, in 1974 only 48 per cent of English-speaking 10-year-olds could answer this question: The milometer on a car shows 0 2 6 9 9 miles. What will it show after the car has gone one more mile?

It is important, however, to record the fact that a very similar question asked by CSMS produced a success rate of 68 per cent among 12-year-olds and that this had risen to 88 per cent by the age of 15 (Hart, 1981). It is encouraging to learn that a very large proportion of British children leave school with an understanding of place-value up to thousands, but many primary-school teachers would be very surprised to realize how many 11-year-olds cannot answer the question, in spite of place-value, on which computational skills for whole numbers are based.

Similar items from assessments undertaken in developing countries were not available for scrutiny in writing this chapter, but a description by Wilson (1978) of the situation in the Caribbean before the inception of the Caribbean Mathematics Project (CMP) in 1970 suggests that the situation may well be no better in some developing countries. For example, 18 per cent was the largest proportion of 12-to-14-year-old children in the territories surveyed who were able to respond correctly, in 1970, to:

$$\frac{2}{3} + \frac{1}{9}$$

Arithmetical difficulties of a similar nature can be inferred from a survey of secondary school pupils in Dakar made by Vandewiele and d'Hondt (1978).

In the developing countries, such problems are often attributed to education systems, which suffer from a massive lack of trained teachers, from overcrowded classrooms and from a severe shortage of textbooks and other resources. While such difficulties are undisputed, consideration should be given to the suitability of the content and the appropriateness of the teaching methods, because the conceptual difficulties cited above are not confined to developing countries. It is to be hoped that the results of the mathematics study carried out in 1980 by the International Association for the Evaluation of Educational Achievement (IEA) will throw some light on the suitability of the traditional primary-school mathematics curriculum in the countries that took part in the survey. However, the youngest age-group studied by IEA was the 13-year-olds, and not many items could be included to sample mathematical topics usually taught in the primary grades.

Much development work has been done, but further studies of the outcome of primary mathematics teaching are needed to assist the evaluation of the suitability of both content and method for the pupils concerned. Such studies need to be locally based. For example, much of the mathematics for very young children in the United Kingdom is based on comparisons: 'more than' and 'less than', 'longer than' and 'shorter than'. In some Papua New Guinea languages, however (and presumably therefore also in children's thinking), this structure does not exist. Bishop (1979) quotes a local interpreter's comment on attempting to translate a mathematics test: 'There is no *comparative* construction. You cannot say A runs faster than B. Only, A runs fast, B runs slow.' However, work in Papua New Guinea has shown that cultural and linguistic differences may have potential for positive as well as negative effects on the learning of mathematics. For example, Bishop found considerable superiority in visual memory among Papua New Guinea students compared to that expected among Europeans. Other fruitful starting points for primary-school mathematics may be available in other cultures in the developing countries. For example, Gay and Cole (1967) pointed out in their study of the Kpelle:

> In Kpelle culture, the local system of measures for dry rice is a perfect beginning for the discussion of the concept of measurement. These measures form an interrelated system, closely analagous to our English system. It is possible to introduce measurement without using this bridge from the traditional culture, but our experience is that the children will neither understand, nor properly use systems of measurement taught in that way. But if the Western units and procedures for measurement are taught parallel with the system the children know, leading the class to see the value of a coordinated, standardized system of measures, the Western concepts will then make sense.

This study was made before the worldwide movement towards the use of metric measures had begun, but the same principles still apply.

Attitudes towards mathematics

However successful or unsuccessful the learning of mathematical concepts and skills may be in the primary years, many other things are undoubtedly learned in mathematics lessons. Among the most important incidentals of such other learning are the attitudes towards mathematics, and the conceptions of mathematics, which the pupils acquire. There has recently been increased interest in the study of this type of unplanned learning.

Most primary-school children in the United Kingdom regard mathematics as useful. The Assessment of Performance Unit (1980) has included questions on attitudes in its surveys of 11-year-old children. It found that:

Most children agreed with statements concerning the usefulness of mathematics.

The mean utility score was very high. From a possible range of 0 to 24, the mean score was 20.4 (SD 3.2), compared with 20.1 (SD 3.7) in the previous year.

Pupils' perception of mathematics as a useful subject is basically independent of how difficult they find it.

The views of pupils were found to be very similar in Dakar, in an education system with a French background. In a survey of secondary-school pupils (Vandewiele and d'Hondt, 1978), 60 per cent of pupils expressed a marked liking for mathematics, giving such reasons as *Les mathematiques sont utiles, Les mathematiques serviront plus tard,* and *Elles sont utiles au pays.*

Mathematics may be regarded as useful by pupils, either for its utility to society or to themselves in their studies or their future work. Alternatively they may see it as a selection device. This latter aspect is evidently not lost on British pupils, as Preece and Sturgeon (1981) found in a survey of 13-year-olds, when one pupil wrote: 'It is helping you to get a better job with a good pay, even though the lessons may be boring and confusing.'

A similar perception of the usefulness of mathematics may be an even stronger result of education in some systems. Gerdes (1981, p. 471), writing about a National Seminar on the Teaching of Mathematics in newly independent Mozambique, records that:

For a substantial number of delegates it was a 'shock' to see that mathematics does not fall from heaven, is not eternal, and is not taught in order to have a mechanism of selection of the pupils, but rather that it has applications.

Other aspects of pupils' attitudes towards mathematics are their perceptions of the difficulty of the subject, and their liking for it.

The development of 'mathphobia' is a worrying phenomenon in both developed and underdeveloped countries. Many adults in both the United States and the United Kingdom display anxiety and fear when asked to do even a fairly simple calculation. In a recent survey of the uses of mathematics in everyday life by adults in England, Sewell (1982) found that:

> There was . . . a widespread reluctance to be interviewed about mathematics. I tried both direct and indirect approaches, I tried replacing the word 'mathematics' with 'arithmetic' or 'everyday use of numbers' but it was clear that the reason for people's refusal to be interviewed was that the subject was mathematics. A church choir refused en bloc, as did some hospital porters . . . several personal contacts also, somewhat to my surprise, were adamant in their refusal. Evidently there were some painful associations which they feared I might uncover.
>
> Those who attributed their fear of mathematics to unhappy school experience . . . mentioned the dismissive attitudes of teachers, and the apparent lack of interest in those who were experiencing difficulty. . . . There were, too, those who dreaded what they saw as the innate characteristics of learning mathematics such as accuracy and speed, as well as the traditional requirement to show all the working neatly. This recalled long-buried anxieties caused by the pupils perceiving the answer by a mental method and being required to produce a written solution demonstrating a method which had not been used.
>
> In many instances, the methods used were of a very simple nature and somewhat clumsy. It appears that a great deal of the arithmetic taught in primary schools has either been forgotten or its relevance has not been perceived.

Similar attitudes are recorded by Nigerian pupils. In a questionnaire administered by Ale (1981), secondary-school pupils listed, among their major difficulties in learning mathematics, fear of the teacher and of the subject, and a poor background, due either to incompetent teachers or to a failure to master basic concepts. There was a general belief that mathematics is for the gifted, and that, no matter how hard one tries, it is impossible to understand it unless one is talented. Moreover, many pupils believed that the African culture is alien to mathematics and to science.

As well as acquiring attitudes towards mathematics, pupils also form conceptions of the nature of mathematics. Probably the best-known study of pupils' surprising conceptions of mathematics is that made by Erlwanger (1973) of the ideas of Benny and of other pupils who were following the Individually Prescribed Instruction (IPI) curriculum in the United States. Benny, in the sixth grade, was doing well in mathematics, proceeding through the individual curriculum at a faster pace than most of his classmates and demonstrating mastery of the work by his regular achievement of over 80 per cent in the tests. However, in a series of interviews, he revealed that he thought that mathematics consisted of hundreds of different rules for different types of problems and that the rules had all been invented. He had discovered that answers could often be expressed in different forms, so that ½ + ¾ can be written as $\frac{4}{4}$ or 1. Only one of the forms was acceptable to the answer key, so Benny believed that his wrong answers were not wrong, but differently written. The rules worked like magic because the answers obtained from applying the rules could be expressed in different ways—ways 'which we think they're different but really they're the same.' It is not only in this curriculum, nor only in the United States, that many children seem to regard mathematics as a game played by arbitrary and incomprehensible rules. Children who have such a conception are totally dependent on the teacher or on the answer-book for judgement on whether they are 'doing it right'. As Sewell relates of an interview with a woman English graduate about her experiences of school mathematics:

> She knew she had learnt all the work by rote and did not understand it at all . . . 'It was all a big con', she said. Her comprehension of numbers has only improved recently with the extensive use of numbers in her job.

Problem-solving

In 1980, the National Council of Teachers of Mathematics (United States) produced its *Agenda for Action: Recommendations for School Mathematics of the 1980s.* The first recommendation was: 'Problem solving must be the focus of school mathematics in the 1980s'. This recommendation was intended to apply to all grade levels, including those of the primary school. Teachers were urged to create classroom environments in which problem-solving could flourish. Students should be encouraged to question, experiment, estimate, explore and suggest explanations. The mathematics curriculum should provide opportunities for the student to confront problems in a greater variety of forms than the verbal format of the textbook.

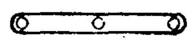
This recommendation followed the poor performance shown in the tests of non-routine problems carried out in the United States in 1978 by the National Assessment of Educational Progress (NAEP) (1979a). While levels of performance on computation with whole numbers stayed fairly stable for 9- and 13-year-olds between 1973 and 1978 (the period of the 'back to basics' movement), the same was not true of problem-solving. And the panel of experts who commented (National Assessment of Educational Progress, 1979b) on the results was 'highly concerned with the generally low performance on problem-solving (application) items and with the declines on these items that had occurred since 1973'.

'Problem-solving' can mean several different things. It may be a simple question of a textbook kind, where the only problem is to discover from the wording of it which algorithm should be used. It may be set in the context of a rather abstract 'real world', when students, for example, use mathematics to compute the probabilities in dice-throwing. Third, it may indicate that attention in teaching is devoted to non-routine problems and the development of the heuristic skills of the pupils. Finally, it may denote that mathematics is being applied to the solution of genuine problems in the children's environment, whether it be the planning of a school journey in a developed country or in the modernization of a village's agriculture in a developing country. Commenting on the 1978 NAEP results in the United States, Carpenter (National Assessment of Educational Progress, 1979*b*) stated:

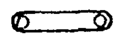
> Children's performance on textbook-like, simple, one-step word problems is pretty good. . . . However, on more complex problems or on problems that deviate even slightly from the textbook format, children have difficulty. It's apparent that many students are not adept at extracting the meaning of a problem situation.

For example, only 3 per cent of American 9-year-olds and 24 per cent of 13-year-olds in 1978 were able to solve the following problem (which, in a developed country, might be thought to relate directly to children's toys):

Mike's building set has:
60 long pieces

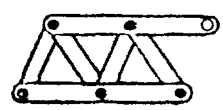


60 short pieces



and 60 nuts with bolts

How many of these can he make?



Placing an emphasis on non-routine problems and developing heuristic ability are foreign to the conception of mathematics teaching held by many primary-school teachers, both in the United States and elsewhere. In the United Kingdom, in the early 1970s, and especially in those schools which were involved in the Nuffield mathematics project, considerable emphasis was placed on developing the children's ability to explore and to discover mathematical ideas for themselves. This approach to mathematics is now rather less prevalent in British primary schools than it was ten years ago, although it is still strongly advocated by those who write about primary-school mathematics.

For example, the HMI discussion document (Department of Education and Science, 1979) *Mathematics: 5 to 11* lists among the general aims of primary mathematics:

An appreciation of the creative aspects of the subject and an awareness of its aesthetic appeal.

An ability to think clearly and logically in mathematics with confidence, independence of thought and flexibility of mind.

An understanding of mathematics through a process of enquiry and experiment.

An awareness of the uses of mathematics in the world beyond the classroom. Children should learn that mathematics will frequently help them to solve problems they meet in everyday life or understand better many of the things they see.

However, learning mathematics through a process of practical experiment and exploration is not the same as building up a pupil's ability to solve mathematical problems in everyday life, or even to solve non-routine problems within mathematics itself. Jeffery (1978) describes the difficulty found by a group of 10- to- 11-year-olds, with whom he worked in the United Kingdom, in attacking an open-ended problem posed in terms of coloured rods of unequal length. The children were asked to make up single colour 'trains', and to find the length of the shortest trains of equal length that could be made with any particular pair of colours. For example, using 4 cm rods of one colour and 3 cm rods of another, the shortest trains of equal length are each of length 12 cm. This situation is rich in opportunities for *making conjectures,* one of the mathematical abilities children need to acquire if they are to become successful problem-solvers. A number of similar heuristic processes can be explicitly learned from the mathematics of the primary years, and children need to employ them for problem-solving. They include:

Making a graphical or diagrammatic representation of a situation, or *representing it by concrete objects* such as bricks or rods, so as to be able to see the mathematics in it.

Noticing *patterns of numbers,* such as the one which appears in the square numbers 1, 4, 9, 16 . . . Continuing these patterns, and, eventually, analysing how they are built up.

Making conjectures such as those embodied in the 'trains' of equal rods discussed above—conjecturing, perhaps, that the length of each of the shortest equal trains is the product of the lengths of the rods used (eg. 12 = 3 x 4).

Explaining why a conjecture is right or, alternatively, *refuting it by means of a counter-example.* Noticing, for instance, that the length of each of the shortest equal 'trains' which can be made with

4 cm and 6 cm rods is 12 cm, rather than the predicted 4 x 6 = 24 cm.

Sometimes, patient *exploration by trial and error* is necessary–to find, for example, whether, in the Konigsberg bridge problem, it is possible to walk round the town crossing every bridge once only.

An important ability for primary children to acquire is that of *tackling problems in discussion with others,* exploring ideas in conversation, and *explaining orally* (and eventually in writing) the progress that has been made towards resolving the problem.

Projects such as the Skills and Procedures of Mathematical Problem Solving (Burton, 1980) have begun to show that such heuristic processes can be learned, and used by children in the 9 to 13 age-group. In the United States, the 1980 NCTM Yearbook, *Problem Solving in School Mathematics,* contains much useful material for primary teachers. However, the difficulties for teacher education, even in the developed countries, are formidable. The experience of Becker, in his abstract submitted to the fourth International Congress on Mathematical Education (ICME IV, 1980), is not uncommon:

> there are difficulties such as: (i) many teachers do not perceive themselves as able to solve problems, or (ii) many teachers do not feel problem solving skills can or should be developed at the primary level.

In many developing countries, with their long tradition of not interrupting or questioning the teacher, and with their overcrowded classrooms, the conversion of primary-school mathematics into a process of active questioning and exploration will be even more difficult.

As Broomes (Unesco, 1981, p. 52), notes:

> the way mathematics is taught, and the materials used for teaching it, have tended to remove mathematics from the conscious minds of certain persons on whom the community depends for its welfare and well-being–the farmer, the agricultural worker and the housewife. . . . Even if the content used to give concrete embodiment of the concepts studied . . . is not part of the body of mathematical knowledge as such, its transmission will serve to forge linkages between mathematics and its contextual use.

An ability to use mathematics in its contexts in the tasks of everyday life and in the development of the economy is of primary importance in the developing countries. It has not always been conspicuously successful in developed countries. Ways need to be found to encourage teachers to take advantage of the links between primary school mathematics and its direct application in the lives of the pupils, their parents and their communities. Perhaps, in some countries, ideas such as those of Gerdes (1981, p. 475) will make for understanding and co-operation from primary teachers:

> Many aspects of the mathematics to be taught in Mozambique reflect a level of technological development which does not correspond to that of the productive forces in the country, but on the other hand, have to be taught in order to beat underdevelopment. How can effective bridges be built from pre-school education and experience to the mathematics to be introduced in the schools? . . . The author thinks that an answer has to be sought, in this phase, in linking mathematics very closely in the first stages of teaching the subject, to applications in the environment of the students e.g., production problems in the new cooperatives and communal villages.

Language and learning

There are considerable language difficulties in the learning of mathematics, even in a country where children are fortunate enough to learn mathematics in their mother tongue throughout their schooldays and where the mother tongue is a Western language, well adapted to the expression of mathematical ideas. When they first enter school, the linguistic skills of many children are insufficiently developed to enable them to join in conversations that have a mathematical content. The school must therefore work to build up the children's concepts and vocabulary. Talk in the classroom needs to stress such phrases as 'the *first* in the family', 'the *oldest* child' and 'the *heavy* box', so that, as children's language develops, they come to understand the mathematical ideas contained in it.

At a slightly later stage, a single mathematical idea will be expressed (if in English) in a variety of different spoken phrases such as: '*count on* 2 from 4', '2 *and* 4 equal', 2 *add* 4', '*the sum of* 2 and 4', 2 *more than* 4'. All these different speech patterns are expressed uniquely in mathematical symbols by 2 + 4. It is not surprising that, if children have insufficient experience of *talking* mathematics, they will later find it difficult to tackle work problems, which can be presented, as above, in a variety of different ways. Talking is, likewise, an essential prelude, if they are to relate their learning to the situation in which mathematics is used in their everyday life. The child's essential difficulty is a linguistic one. The mechanical working of 2 + 4 is not difficult. But if the concept and symbolism of addition have not attached themselves to the range of spoken phrases used by the teacher and the textbook the child will not be able to arrive at the symbolism 2 + 4.

In the teaching of reading, the first aim is to help a child to make a correspondence between the written symbols he sees and the sound and meanings of the oral language, which already makes sense to him. In the teaching of mathematics, and in its reading and writing, the difficulties are compounded. This is partly because the corresponding oral language is not always meaningful when the written language or the mathematical symbols are introduced and partly because a single set of mathematical symbols corresponds to such a variety of oral language.

In the United Kingdom, and to some extent in other developed countries, the use of individualized learning schemes for mathematics in the primary school has increased recently. Such schemes make considerable demands on the child's ability to follow written explanations and on his ability to learn mathematical concepts by abstraction and by generalization from activities communicated to him through written instructions. In classrooms where individualized work is in progress, children may not have sufficient opportunity to discuss mathematics orally either with their classmates or with their teacher. In such cases, written language may hold an importance for which the children are not yet ready. The growth of this style of teaching has focused interest on the problems of using written materials in mathematics.

A recent production by the BSPLM Language and Reading in Mathematics Group (Rothery, 1980) has drawn attention to the range of different purposes which mathematical text can serve and the problems of vocabulary, syntax and symbolism the child may encounter in his reading. Text can be used for a variety of different purposes. They include the exposition of mathematical concepts and skills, giving the child instructions to write, calculate or undertake a practical activity, and supplying examples and exercises. In the traditional use of textbooks, when the mathematical ideas are expounded by the teacher in a class lesson, the child only needs to understand the last of these forms of writing. Even when text is used in this traditional way, many children are only able to cope with the reading of calculations expressed entirely in symbols. They seem unable to visualize the situation represented by a word problem. So they confine their reading largely to picking out the numbers in the text and they use 'cue words' to help them to decide what operation to perform on the numbers they have picked out. When text is used as a major resource for individual learning, children have to tackle a greater variety of reading problems. But this additional experience of reading mathematics does not necessarily produce greater reading skill, unless the teaching is specifically directed to reading for learning.

The vocabulary of a mathematics textbook also contains pitfalls for the unwary. There are some words that are used only in mathematics—words, for example, like *parallelogram* and *hypotenuse* (Rothery, 1980):

> These words are encountered only in a mathematical context and their meanings must be learnt from the mathematics book or teacher. . . . Once a word has been forgotten it is not easy for the child to find out the meaning unaided. Furthermore,

such technical words are often of key significance and failure to understand them can lead to total failure to read the passage.

There are also many words that carry different meanings in mathematics from those they have in ordinary usage. For example, in English, the difficulty of teaching the concept of 'difference' to young children is compounded by the fact that the word 'difference' is used in mathematics to denote one aspect of the idea of subtraction, whereas the child will previously have used it in everyday life to denote one of a variety of dissimilarities, rather than only differences of number. The BSPLM group has now started to study how writers may devise text that would be more readable by the pupils, together with ways in which the pupils' reading ability in mathematics may be improved and methods of helping the teacher to use the text more effectively as a resource for learning.

In countries where children learn mathematics in the primary school in a language other than their mother tongue, where the language used changes as the pupils progress through the school, or where the mother tongue is not well adapted to the expression of mathematical ideas, the difficulties are even more complicated. Much useful information is contained in the report of a Symposium on Interactions between Linguistics and Mathematical Education, sponsored by Uncesco, the Centre for Educational Development Overseas (United Kingdom) (CEDO) and the International Commission on Mathematical Instruction (ICMI) in Nairobi, in 1974 (Symposium on Interactions . . . , 1974). The theme of the symposium was largely characterized by the adaptation to mathematics of a quotation by Dart about science teaching (*Physics Today*, June, 1972):

> The earliest and most important stages of a child's education begin . . . with informal imitative play. . . . This . . . is the world's largest and best school system. It has more students and more teachers than any other, it enjoys a more favourable student-teacher ratio and has more class hours than any other and is by far more effective than any other school system known. . . . When the first formal schooling is added to this informal instruction, it makes a profound difference whether or not its teaching is consistent with the informal or at variance with it. This becomes a matter of considerable importance where a foreign system of thought is to be taught by means of an imported school curriculum, as is the case with science in so much of the non-western world, and even in some parts of the United States.

Many instances are quoted in the report of the symposium where the mother tongue presents problems in choosing suitable vocabulary to express mathematical ideas, even though language forms may be available in which to express the concepts. For example, many African languages are rich in words denoting various collections. But in Igbo, a language used in parts of Nigeria:

> The word 'otu', literally meaning a band (eg. of robbers; a group of old men, young maidens etc.) is normally used to refer only to human beings who have something in common. It is this latter word . . . that teachers of modern mathematics in the elementary schools are using for 'set'. This usage is slightly misleading, or at first confusing, to the children, because it suggests to them that elements of sets can only be human beings.

However, there may be strengths of the mother tongue which make for mathematical opportunities, provided these strengths can be identified, and the curriculum developed in such a way as to take advantage of them. For instance:

> English is a sloppier language than Kpelle. English uses the same word 'or' sometimes to mean 'either of the two', sometimes to mean 'either of the two, but not both', and sometimes to mean 'in other words'. In Kpelle no such ambiguity can arise. . . . This led to the speculation that Kpelle subjects would probably perform more efficiently than American subjects a test of ability to learn logical rules—a speculation subsequently demonstrated and confirmed.

When the language of schooling from the beginning of the primary school is a second language, there is less possibility of developing mathematical ideas through informal oral language before the written language and mathematical symbols are introduced. It is vitally necessary that the language teaching should be designed in collaboration with the mathematics teaching, so that language is available to express mathematical concepts as they are developed:

> The linguistic concepts and structures have to be taught. And if they are to have meaning they must be taught in circumstances which simulate the day to day situations which arise naturally in the home. . . . This means that mathematics must not be taught by the teacher writing symbols on the blackboard, rearranging them, getting 'answers', asking the class to copy the process and to learn it by heart. Instead the teacher must be trained to involve the children in carefully structured activities, investigations and discussions which will ensure understanding. In short, the teaching of mathematics in a second language must, in effect, adopt the principles which govern the methods of teaching a second language as a language.

The principles of developing meaning in the teaching of mathematics to young children are the same, whether the child learns mathematics in his mother tongue or in a second language. They are very clearly expressed in the above quotation from the report of the Nairobi symposium, and mathematics would be more effectively learned if these words were better understood by many teachers who work in more favourable conditions.

Sex differences

Sex differences in the learning of mathematics have been much studied in the United States, and to some extent in other developed countries. In the primary years, boys and girls are already developing different attitudes to mathematics and its learning. Boys more often expect that they will find mathematics useful in their future work. Girls are already beginning to display the lack of confidence that will be so damaging to their learning of mathematics in the secondary school. For example, the Assessment of Performance Unit (1981), in an attitude test given to 11-year-old children, found that:

> Significantly more boys than girls believed that they usually understood a new mathematical idea quickly, that they were usually correct in their work and that maths was one of their better subjects. In contrast, at least 9 per cent more girls than boys (a statistically significant difference) confirmed that they often got into difficulty with maths and were surprised when they succeeded.

There is also some evidence that not only differences in attitude, but also differences in attainment in mathematics, start in the primary years. Such differences are not global, but relate to the different areas of mathematics in which boys and girls perform well. The APU surveys found significant differences in performance in several content categories. In 1978 (Assessement of Performance Unit 1980), girls did significantly better than boys in computation (whole numbers and decimals), while boys were significantly ahead in the 'measure' of length, area, volume and capacity, as well as in applications of number and in rate and ratio. In 1979, there were two additional categories in which boys did significantly better. One was the 'measure' of money, time, weight and temperature; the other the category of concepts (fractions and decimals). Shuard (1981) analysed the differences of performance between boys and girls in a test given by Ward (1979) to 10-year-olds in England and Wales. The results suggested that girls were ahead in computation by the age of 10, while boys were ahead not only in work on measurement and items involving spatial visualization (both of which are well-documented in the literature), but also in the understanding of place-value, a concept that is central to understanding and future progress in mathematics.

In the United States, however, sex differences in mathematical performance in the elementary grades seem less clear. Fennema (1974), analysed studies of children aged 10 to 14. She concluded that:

> Girls performed slightly better than did boys in the least complex skill (computation) . . . In the 77 tests of more complex cognitive skills (comprehension, application and analysis) five tests had results that favoured girls, while 54 tests showed significant differences in favour of boys.

In the developing countries, little evidence seems to be available about specific differences in mathematics learning between the sexes. In the United Kingdom, however, boys perform better than girls in public examinations in mathematics at the age of 16, and more boys than girls choose mathematics or related subjects for study after the age of 16. A similar picture seems to obtain in the United States.

The 1980 IEA survey has studied sex differences in mathematical attainment (Steiner, 1980) noting, in its preparatory work, that:

> The relevance of different variables in explaining sex differences may depend on very broad environmental conditions which may differ from country to country. . . . The data analysis should be carried out separately for each country, to try and see if the interaction between variables is the same in all countries, or to find out what differences exist.

The results of this study have not yet been published.

Implications for teacher education

These contemporary problems and trends in primary mathematics have immense implications for pre-service and in-service teacher education in both the developed and developing countries. If the goals of the primary mathematics curriculum are based on encouraging children to think mathematically, rather than on encouraging the learning of computational processes by rote, these goals are not, at present, consonant with the personal goals for mathematics teaching of a majority of primary-school teachers. This is true even in a country as highly developed as the United States, where a 1978 survey (Denmark and Kepner, 1980) of opinions on 'basic skills' among NCTM members and their colleagues showed that:

> Forty per cent of the elementary (K-6) and college teachers indicated that the basic skills should be taught before the introduction of underlying concepts and applications; 61% of secondary teachers agreed with this instructional strategy.

Nor is the trend towards a greater emphasis on the use of the calculator in primary schools generally acceptable either to primary teachers or to the public. In the survey mentioned above, only 32 per cent of the elementary teachers thought that knowing how to use a hand-held calculator was a basic skill. Many primary-school teachers commented that calculators should only be used *after* students had become proficient with pencil-and-paper algorithms.

The trends also indicate a need to develop new teaching styles that emphasize exploration with concrete materials, mathematical conversation, problem-solving and the application of mathematics to the everyday life of the pupils, often outside the classroom. The conception of primary mathematics teaching held by many students on entering pre-service teacher education does not include the use of these teaching styles, especially in countries where there is a strong tradition of learning by passive reception, and where overcrowded classrooms make more active learning extremely difficult for teachers to implement. Moreover, when they visit schools, students do not often see innovative teaching styles in operation. So they have few, if any, models on which to base their work, and little evidence about the effectiveness of the new styles.

It cannot be expected that students or teachers will change in fundamental and radical ways unless they have a powerful inner conviction that the new methods will be more successful and more rewarding than the methods they replace, and unless they see success beginning to emerge from their first tentative efforts to change. Pre-service teacher education alone is most unlikely to be successful in producing lasting changes of attitude in students. This is shown by experience in the United Kingdom and the United States, where it is quite usual for courses in methods of teaching mathematics to emphasize trends such as those described in this chapter. The PRISM survey, however, indicated that teacher educators and mathematics supervisors held much more radical views about mathematics teaching than did the classroom teachers. Teachers, in their pre-service training, come under the influence of these radically-minded teacher educators, and, after finishing their training, are supported in their classroom work by mathematics supervisors whose views are similar to those of teacher educators. But the conservative climate of the schools, and the attitudes of parents and the community, often have more influence on teachers than do innovative educational views.

It should not be expected that teachers will effectively change their styles of teaching if they themselves learn of new methods only by reading or by passive reception. A new curriculum, whether centrally devised and universally implemented, or chosen by a school when it buys a new set of textbooks, does not, of itself, produce changes in the teacher's approach. The child may say to his teacher, 'The book says I need the tens-and-units blocks to do this.' But, even in a well-equipped school, the teacher's response may be: 'The blocks are in another classroom. There's a picture of them in the book—you can do it by thinking—you don't really need the blocks.'

Experience in the United Kingdom suggests that teachers who are themselves involved in groups devising curriculum guidelines and activities are much more likely to exemplify, in their own teaching, the beliefs which have developed in the group than are teachers who merely read the guidelines, or who operate the new curriculum without extensive dicussion of its aims, content and methods. The same effects upon teacher involvement emerged in CMP (Wilson, 1978):

> The CMP set itself, as its major objective, to improve both the professional competence and the personal confidence of teachers of mathematics. . . . It largely succeeded in this. 'I have taught mathematics for 20 years, but these last two years have been the most enjoyable, the most interesting, the most challenging–because of this project' is a not untypical comment. Many showed a totally new attitude to mathematics as a result of being given responsibility for the development of teaching materials, even on a small scale. More generally, there is a much greater awareness among teachers in the region of the nature of the difficulties faced by children learning mathematics, and of how the teacher can try to overcome them. . . . Project teachers have developed a much freer and more versatile style of teaching, and the introduction of worksheets, group-work and a greater use of simple teaching aids has had its effect in other subjects. . . . Teachers were trained in evaluation techniques, including those of diagnostic testing of pupils' difficulties, and of the effectiveness of a particular teaching strategy in overcoming them.

As in all innovations, success was not total, and not all teachers espoused the new methods with conviction. Circumstances changed. The Project came to an end, and many teachers reverted to previous methods:

> Its success was marked with some teachers, but it failed with others. In the case of the most fully involved and interested teachers, the work of diagnosis and prescription still goes on. In a remote primary school in Dominica, high in the hills, the principal conducts a short weekly seminar for his staff (sending the children home early for the purpose), and is himself devising simple teaching units which are duplicated on the school machine. . . . It is unfortunately more common to find less self-confident or adventurous teachers reverting to the use of a more traditional textbook, and therefore to the course and syllabus which it dictates.

However, the reversion is never complete. Parts of the new thinking are absorbed into the general educational climate. The next stage in the continual process of curriculum renewal and development does not start from the same position as did the previous stage.

The major implication for teacher education is that it cannot proceed in isolation. Curriculum development and pre-service and in-service teacher education must all go hand-in-hand if they are to have any real effect on the teaching of mathematics. Only thus can all the available workers support one another in encouraging teachers to espouse new goals and to work towards their implementation. Moreover, schools do not work in a vacuum, isolated from their local communities. Many parents take a deep interest in their children's education. They may supplement it by their own efforts, or undermine it by their criticism. They may even contribute to the school by fund-raising or by their own voluntary help in the school. Mathematics is a subject that produces much anxiety in adults around the world. It is important, therefore, that parents should, as far as possible, be involved in understanding what is happening in their children's mathematics learning, and that their approval should be sought for changes of practice. The involvement of agencies outside the school, such as those concerned with curriculum development and teacher education, may help in gaining the understanding by the local community and by the parents of the primary school's work in mathematics.

Both pre-service and in-service teacher education should help teachers to understand their own professional growth and to monitor its progress as they learn to use a greater variety of teaching strategies than those with which they started their careers. In every class, there are children who do not learn mathematics as effectively as their ability suggests they should, but many teachers still lack the skills needed to diagnose the reasons for children's failures and to treat them in a way that is consonant with each child's mode of thought. It is necessary to strengthen the training of primary teachers in the arts of diagnostic conversation with children of designing appropriate experiences which are based on children's present understanding, and of analysing the results.

Primary-school teachers not only teach mathematics. In most countries, they teach across the entire curriculum. Their skills in mathematics teaching are related to their more general teaching skills. In pre-service teacher education, it is necessary for mathematics-teacher educators to work in collaboration with other teacher educators to ensure that the short time available in pre-service courses is effectively used to develop positive attitudes in students towards active learning and thinking across the primary school curriculum. Similarly, in in-service education, there is scope for work which integrates mathematics into the teacher's total curriculum thinking and planning.

In this chapter, the need of primary teachers to feel confidence in their personal knowledge of mathematics has not been discussed. But it is necessary for pre-service teacher education to do all it can to ensure that beginning teachers have a deep enough understanding of mathematics to be able to transmit ideas correctly. So a knowledge of mathematics also needs to be integrated into the unified education of a primary teacher. As Freudenthal (1978) wrote about teacher training in the Netherlands:

> Change in teacher training at the primary level requires a far-reaching integration of teaching subject matter with its didactics—and, as a precondition for the last: that the subject matter learned by the future teacher is so close to the subject matter to be taught by him in his future career that it is accessible to this complete integration with didactics. . . . Developing a mathematical attitude as a means of developing a good educational attitude is judged to be more important than the quantity of mathematics taught to the students. Changing teacher education in

just one subject—mathematics—looks like a quixotic adventure; but somewhere you have to start, and might mathematics not be the easiest point of departure?

Any 'wrong knowledge' with which a student enters teacher education can the more easily be corrected if, at the same time, the student is learning to believe that children can think mathematically, with the implication that the teacher will also be able to think mathematically. If the student is developing an attitude of positive exploration in the teaching of mathematics, a teaching skill is being learned that will be greatly needed in coping with the range of curriculum innovation in primary-school education that is inevitable in all countries in the next decade.

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Fidel Oteiza

The environment as a source for the elementary mathematics curriculum

Introduction

This chapter attempts to analyse the possibilities as well as the limitations of students' everyday environment as a source for their mathematics curriculum. It takes as its starting point the following premise: 'The environment has always been a powerful stimulus for mathematics activity.' From this, several elementary educational implications are deduced. The paper next stresses the importance of formal thinking, thus completing the survey of educational implications. It then goes on to show how different curricular approaches interact with the school environment in particular ways. The question of values is raised, as well as some related questions concerning the ways in which different schools of contemporary thought approach the environment. The chapter then considers the problem of the selection of objectives and the inevitable question: what mathematics does the adult who has left school require? An answer is offered in the form of a choice between the alternatives examined in the preceding sections.

Next, on the basis of experiments carried out in Latin America, a special procedure is described for selecting the objectives of elementary-school mathematics. From the results of these experiments some guiding principles are deduced, and a number of methodological recommendations are put forward. The study concludes with an attempt to determine the implications for teacher training of the approach described.

The environment has always been a powerful stimulus for mathematical activity

It was no accident that mathematics, a creation of the Greeks, initially took the form of geometry, with its description of shapes and relationships belonging to space around us. We now know that there are an infinite number of possible geometries. The doors were opened to this plurality by Lobachevski and Riemann as recently as the last century.

But Euclid's geometry is the one that uses concepts devised from sensory experience. This is because the senses are the source of the images upon which are built such concepts as the point, the line, volume, parallels, intersections, and so on.

The many models that have been developed to describe, explain or predict physical phenomena are also mathematical creations. The concept of a vector enables us to handle the interactions of physical phenomena, such as movements, accelerations and forces. The birth of the infinitesimal calculus was—at least with Newton—closely linked with the search for a model of changing speed under the influence of a central force. A coin tossed in the air and a stack of playing cards provided Bernoulli with the stepping stones into the now enormous world of probability theory. Likewise, Lagrange, Laplace and Hamilton developed a remarkable range of mathematical models to permit the solution of astronomical and mechanical problems. In the present century, statistics and informatics undeniably represent an applied form of mathematics. Also, inventories, censuses, experiment at work and data processing are all subjects involving the relationship between man and his environment.

The above examples should suffice to illustrate the basic affirmation, namely, the environment is a rich source of stimulus to mathematical creativity. From this premise, we can deduce some methodological implications for elementary mathematics and for its environmental dependence. They are:

The mathematical concepts and models to be derived from the environment are numerous and should be sufficient to occupy fully the time of any student and meet the requirements of any school.

Mathematical concepts and models are not readily discerned within the physical, chemical or social phenomena that give rise to them. It is therefore necessary to make use of the environment in a way that enables us to proceed from the particular to the general, and from the phenomena to the laws that govern them.

It is important to examine the ways in which researchers (who develop mathematical models) have approached the environment and use their techniques as a basis for the development of teaching methods.

It seems important to account for the humble origin of many valuable mathematical models, since it is difficult to discover their origin in the finished formulae contained in books. Thus, information relating to origins is not readily available to the teacher responsible for introducing young people to the understanding and use of models.

Last, we should remember that the discovery or the construction of a model is something that is highly inductive. It is characterized by forward movement and back tracking, by sudden changes of direction and by a measure of luck on the part of the individual conducting the experiment. At their birth, models are lacking in general, formalized characteristics, and often produce only pseudo-proofs. It is useful to keep this in mind when conducting teaching experiments whose purpose is to mathematize aspects of the environment.

The contribution of thought

It is a long way from the fluxions of Newton to the formulations of Weierstrass, even without taking into account the two centuries separating them on the time scale. Where is elementary intuition to be found in the theory of groups or in a classification of topological spaces? While the environment is a powerful stimulus, mathematics is nonetheless a creation of man. Concept upon concept, structure upon structure, man has built a complex and powerful edifice of reasoning. And it is primarily this theoretical construction that we call mathematics. History shows that it is not only the environment, but also the models to which it has given rise that provide stimulus for mathematical creativity. On the long journey between the first intuition, sparked by the observation of nature, and the formal models of phenomena, structures successively destroy one another, making way for the development of other more abstract, more general or simpler ones. History shows, too, how some of these abstractions, seemingly remote from reality (e.g. complex numbers, groups or tensors) seem to have been awaiting the opportunity to be transformed into applied models. The theories underlying telecommunications, quantum physics and relativity itself are used in the expression of laws or in the description of phenomena.

It is precisely this ability to theorize that distinguishes human intelligence. A methodological approach would not be complete if it did not prepare the student for deductive work. The environment can be the starting point, or it can be the point attained, but it is the mutually transforming interaction between man and his environment that will enable the student to structure his intellect by structuring reality (Piaget, 1971).

The formal nature of mathematics also has implications for the construction of methodological approach. They include:

It has its own linguistic register which must be learned and used. What should be required of this language, and how can it be incorporated into the student's repertory?

It is a formal language. The student's training should acquaint him with graphic illustrations and the symbolism.

It implies abstraction and generalization. The search for regularities, classification, the definition, the definition of general terms, the induction of formulae or of models from specific cases, all contribute to the development of these capacities.

The construction of formal mathematics presupposes the development of perfectible models. This construction has a dynamic aspect that should be present in the teaching, just as the relativity of models should always be kept in mind, since they can be improved.

Rather than continue this list, we may conclude by stressing the importance of developing in the student the capacity for inductive reasoning. The use of the structure 'If A, therefore B' should be frequently and carefully applied. This is another way of developing the student's powers of axiomatic reasoning. It helps him to achieve economy of assumptions and absence of internal contradictions in the models he uses.

These considerations raise a number of important issues. How can we use the wealth of the environment, as a generator of mathematical thought, in the training of mathematical talent? How can we significantly relate the environment to the formal body of mathematics so that the student can benefit from both sources? What sort of training brings the teacher to recognize abstract structures in the environment that surrounds him? How can he be helped to use this recognition in organizing the encounter between his students, the environment and mathematical models? In order to tackle these questions, it would seem necessary to develop procedures for establishing an appropriate nexus between school mathematics and the environment.

Curriculum trends

In order to evolve a mathematical curriculum from the environment, it is necessary to give preference to one source of educational objectives over others. R. Tyler (1969), in his well-known model, indicates three sources for school objectives: culture, society and the students themselves. In this framework, the environment is defined in terms of the achievements of mankind (ie. culture), the requirements of society and the needs or aspirations of the students. The source of the objectives is also linked with the procedures used to detect, select and formulate them. Traditionally, the mechanisms through which objectives were incorporated into teaching were: the education tradition itself, including traditions of distant countries, such as those followed by many Third World countries that have cultural links with others; gradual transformations brought about through practice and induced by changes coming from outside the school; reforms decided upon at a higher level of policy; and—least frequently of all—changes founded on diagnostic studies or on assessment of needs.

Both the origins of the objectives and the procedures used in selecting them are influenced by philosophical and value-related options of a more general nature. According to their respective philosophies, McNeil (1977) identified four contemporary curriculum styles: a humanistic style, an academic one, one directed toward social change and one that is primarily technological.

This classification—although arbitrary, like all such classifications—is helpful in pointing out certain basic issues. How do different education currents perceive the relationship between the student and his environment? How do they make it effective? A curriculum may be termed humanistic if it focuses attention upon individual achievement (Oteiza and Messina), and if it encourages the development of liberating forces oriented towards providing the student with integrating experiences. Creativity and self-reliance are the two specific aims of those who adhere to this kind of approach.

A curricular style will be called 'academic' in this classification if it is centred on knowledge and if, basically, it promotes the development of cognitive structures. Here we may readily recognize the reform movement of the 1960s, and it is sufficient, perhaps, merely to recall the report of the Cambridge Conference (1963).

A curriculum can be classified as sociological or oriented towards social change if its social requirements take precedence over personal needs. In the view of Paulo Freire (1970), the role or purpose of education is the transformation of the world through the transformation of consciousness.

Finally, a curriculum may be classified as technological if its action is directed towards the development of effective and efficient teaching systems. Under this approach, the curriculum is planned along careful lines and is susceptible to continuous improvement or optimization.

Each approach implies an option in terms of philosophy and values. Each approach also handles the relationship between man and his environment in particular and produces educational objectives from its distinctive standpoint. The humanist sees the environment as a source of personal fulfilment. The relationship between the student and his environment should take place in a permissive atmosphere. The environment will therefore be a source of objectives or the subject of applied actions, to the extent that the student derives benefit from it for his own personal training. Experimentation, observation and the application of knowledge—these are the three activities encouraged by the humanistic approach.

In an academic curriculum, however, the situation is different. Here the source of educational objectives is culture. The subjects point out the path to be followed. Man approaches nature or his environment in order to gain understanding and put his hypotheses to the test. Here the relationship between the environment and the mathematics curriculum is a distant one. However, the formal aspects of mathematics become important.

For those who see the curriculum as an instrument of social change, the relationship between man and his environment is of fundamental importance. There would be no human action if there were no objective reality, a world external to and capable of challenging man's ego (Freire, 1970). What is important here is active interaction—'praxis'—which is 'reflection and action of men upon the world in order to transform it' (Friere, 1970, p. 36). The relationship will in this case be a dynamic and reflective one, with the environment challenging man; a call to action, with man transforming the environment and reflecting on the meaning of his action, thereby modifying his own knowledge.

From the technological standpoint, the environment must be structured so as to create external conditions that will help the learning process (Gagne, 1977). In this case, the development of 'means' replaces the direct approach to the environment.

As was mentioned earlier, this classification of contemporary curriculum styles is arbitrary and far from all-inclusive (Eisner and Wallace, 1979). From the standpoint of the present study, what is important is what underlies the discussion. Different philosophical concepts means different ways of visualizing the environment and, therefore, different forms of interaction between the student and his environment. Conversely, the use of the environment as a source of objectives and—especially—the choice of a particular way of using it for this purpose means opting for a particular set of values. A corollary of this is that the teacher must be prepared to make this kind of choice. And he must have the theoretical knowledge that will enable him to understand his action.

It would be interesting to trace the causes of the current concern with the environment as a source of objectives. Might it be nothing more than a reaction to the excessive academicism of the 1960s? Just another swing of the pendulum of 'fashions' followed in education? Or is it indicative of a more serious choice reflecting concern about the conditions in which millions of human beings are living? Our lack of historical perspective limits us to merely raising these questions. In any case, the trend implies dissatisfaction with the existing curriculum. And, in its concern for applied, environment-oriented teaching, education is abandoning an elitist posture that leaves it to the wise men to decide which mathematics is right for the people—and bringing part of the decision-making process into the classroom.

Different styles and approaches

Before we analyse the methodological implications of the approach, it may be useful to summarize some of the conclusions to be drawn from the previous sections. These are:

The student's environment is a valid and rich source of starting points that can be used in preparing a mathematics curriculum.

The mathematization of the environment, or the creation of abstract models based on an analysis of reality, is a complex task which requires training and knowledge.

It is important to determine which are the skills a teacher needs in order to be able to recognize mathematical structures in the environment and to use them in the preparation for completion of the curriculum.

The development of mathematical models is usually an inductive exercise, characterized by trial and error, by back-tracking, by re-formulation and by changes in direction. These characteristics are in sharp contrast to those to be found in classes of an expositive type.

The development of formalisms and theories is an important part of the construction of the mathematical edifice. This 'second moment' of mathematics (Dienes, 1966) is axiomatic, deductive, formal, symbolic and remote from any physical or material references. It seeks internal consistency as well as economy of assumptions. In the development of a teaching method, the presence of this 'moment' is important. Consequently, in any attempt to use the environment as a source of mathematics, care should be taken to specify the manner in which the exploratory, applied, empirical process will be accompanied by the necessary reflection and formalization.

Lastly, the choice of objectives and procedures is an exercise in which values play a major role. On the one hand, this means that any use of the environment for purposes of school mathematics places an important responsibility on the teacher. On the other hand, the very term 'environment' has quite different connotations, according to the philosophy of the person using it.

What mathematics do adults need?

What mathematical knowledge does a city dweller, a peasant, or other adult, who has left school, need? There, we cannot afford to generalize. Schools have always asked this question; so they teach elementary operations, decimal numbers, fractions, ratios and percentages, among other topics, precisely because they consider all these necessary for daily life. The question must be asked in a different way. The ability to recognize a particular subscriber in a telephone book, or a relationship in a family tree, or the scale of a map or of a model are not unimportant skills. We will return to them in the discussion of method. But is it not possible to go further? The questions that guided the experiments on which the present study was based were the following:

What mathematics do these particular men, and these particular women, living in this particular community need, and because of what particular problems do they need the mathematics?

What mathematics will enable them to better handle their work, to understand the structures of the relationships involved and to organize their community or their life?

Which models, in which order, and constructed in what manner, can best contribute to the training of a young person deprived of stimuli?

What strategies for the solution of problems do they possess, and which can be better taught with a greater probability of their being applied to other problems? (Oteiza, 1977).

To answer these questions, it was necessary to evolve a procedure for selecting mathematical objectives, based on an analysis of requirements. Its main features are set out below. The experiments in which it was used gave birth to a programme designed for adults. The programme was used for teaching very different groups of persons: adults attending evening classes, workers in self-management enterprises, fishermen from several coastal communities and a variety of groups of organized workers, including some from the rural sector (Montero and Oteiza, 1977).

It was this very diversity that forced the design of a programme that would be both adaptable and flexible, and applicable.

To determine the objectives, we proceeded to work with the groups concerned. Why did they want a teaching programme? What did they expect of it? What mathematics did they need, or think they needed? The answers given were as diverse as the groups in need: 'We need to learn accounting'; 'I have to be able to read a balance-sheet'; 'We need to be able to read a blueprint'; 'Look at these metal parts, they're made from these diagrams'; 'We have to be able to do it ourselves'; 'We want to be able to understand these figures' (instruction tables for fertilizers and for chemical treatment of some animal diseases).

Another type required was connected with obtaining school certificates: 'I must pass the sixth grade exam to be able to continue my work'; 'I want to finish elementary school'; 'We want to go on studying'.

In order to cope with such a diversity of need, it was necessary to develop a model that at one and the same time would be specific to each group, applicable to several groups and suitable for all groups. Accordingly, three concepts were used: the specific curriculum, the applied curriculum and the minimum curriculum. They were obtained by superimposition. By looking, in turn, at the specific needs of each group, sets of objectives $0_1, 0_2, 0_3, \ldots$ are obtained. Now, if $0_1, 0_2, 0_3 \ldots$ are superimposed, we shall find some items common to all sets. Some will be specific to each group. The name *specific curriculum* was given to the objectives that did not intersect with any other set of objectives. The specific curriculum was needed only by the particular group concerned. The name *applied curriculum* was given to any set of objectives that belonged to two or more sets of objectives, but not to all of them. Finally, the name *minimum curriculum* was given to the set of objectives that belonged to all sets.

By definition, therefore, there are as many specific curricula as there are different groups of individuals needing to learn mathematics. There is only one minimum curriculum. And there will be several 'applied curricula'.

As examples of objectives belonging to each of these three types of curricula, the following may be cited:

Specific: 'The interpretation of scales' was an objective of only one group of adults.

Applied: 'Proportions in mixtures of liquids' was an objective of several of the groups.

Minimum: 'Operations with decimal' numbers was an objective present in all the curricula.

The study began, in each case, with the specific curriculum. Once the objectives necessary for a particular group had been determined, we proceeded to deduce—through a working analysis—the means of attaining them. Next, a different strategy was used. While the purely utilitarian character of the participants' requirements was taken to be an acceptable criterion, the question arose whether the utilitarian aspect was sufficient. It was decided that utility was not enough. Instead, it was thought desirable to analyse the stated needs in the light of the following principle. Participants should be equipped to study more advanced forms of mathematics or to study technologies they can incorporate into their work. The aim here is to promote the independence of the student through high-level cognitive strategies (Gagne and Briggs, 1979). So, in considering each topic or each set of objectives, the following questions were asked:

What else can be done with the same contents? In a horizontal direction, ie., in application to new situations, and in a vertical one, to gain acquaintance with higher level concepts?

Will the mathematical language used enable him to carry his studies further if he so desires?

Which cognitive strategies implicit in the objectives can be further developed?

As a result of this second analysis, we incorporated into the curricula derived from the consultation process applications, elements of modern mathematical language, complementary geometrical work and—mainly—problem-solving. In several cases, complementary material was produced in which higher level, formal or symbolic aspects were introduced. The specific curricula were, therefore, the result of an empirical study and a theoretical and technical analysis.

The model was also used to determine teaching activities, and several criteria for selection were employed. For example, the specific curriculum concentrated on case-solving, group work, simulated games and problems drawn directly from situations encountered in life. But in the applied curriculum a broader spectrum of examples was used. In some cases, real situations were drawn both from agricultural and industrial sources. On the other hand, the examples used to teach the minimum curriculum were drawn from everyday working-life. Similarly, work was done on graphs accompanying written texts, on the form of Spanish used, and on factors that could be sources of motivation.

The model revealed a power to pull new groups into the programme. The fishermen, for example, wanted to learn the elements of geometry for the purposes of boat-building. They also found a need for financial mathematics. Their specific curriculum—the fifth one developed—was a small one, and, for the most part, they were studying measurements and the reading of blueprints. What they had already studied in the area of accounting could be used by them, with slight modifications.

A specific but universally flavoured programme was thus produced. While it could be used for solving practical problems within a well-defined area, it also opened new possibilities to participants. It even opened the path to movement from one curriculum to another simply by studying the complementary matter in addition to what had already been learned.

To sum up, therefore, as many specific curricula were produced as there were groups needing to learn mathematics. Each one of these curricula was the result of an empirical analysis carried out from the standpoint of the environment and of a 'theoretical' analysis carried out from the standpoint of both mathematics and educational psychology. The intersections of these curricula provided the minimum curriculum and served as a guideline for responding to new groups of adults interested in participating in the experiment.

Methodology and stategies of education: some guiding principles

Operative aspects are crucial in any educational approach. In the present case, that of the development of an environment-based mathematics curriculum, it is necessary to suggest certain methodological principles. We describe below a methodology in keeping with the premises arrived at in the preceding sections, by means of axiom-like statements which might provide guidance for specific action. The reader will notice, in this section, the influence of the Swiss psychologist Jean Piaget; several of the recommendations made are based on his work.

General recommendations

Regarding the selection of objectives, the following elements should be added to the basic model referred to in the preceding sector:

The curriculum should be the result of the needs and aspirations of the participants. If the latter form an organized group or have strong cultural links (e.g., are members of a community), the development of the curriculum should begin with a participatory research project. In this, the group and the researcher together seek to define the educational goals which will best help the members to achieve their purposes as a community.

In this sense, the start of the process of developing the curriculum involves both the participants and the facilitator in the learning process.

However, it is necessary to go beyond purely utilitarian objectives and to serve as a bridge towards the attainment of new knowledge, in addition to teaching the participants to study independently.

Special attention should be given to the difficult task of facilitating the learning of technologies or the use of models which are external to a culture, without destroying the participants' self-confidence or their belief in their own community's resources. The participants confront the mathematical task with 'reasonable but incomplete' strategies, as Robbie Case (1981) put it. The technique of discovering spontaneous strategies and then constructing with them proved fruitful in our experiments. It also yielded excellent results for Case.

The following may be regarded as guiding principles for learning experiments (Messina and Oteiza).

The principle of construction

The teaching should offer the student activities that will enable him to develop or rebuild his knowledge. These activities should be carried out by the student in direct relationship with his environment. It is this environment which he will subject to research and transformation.

The principle of organization

Students should be encouraged to create systems for organizing the information acquired. The teacher should help to organize the interaction between the student and the environment and then facilitate the oral or written expression of what has been observed and studied. This will open the way for the development of organization schemes, resumés, diagrams, tables, flow-charts and relationship diagrams.

The principle of the functionality of knowledge

The teaching should lead to the creation of structures that reflect an increasing degree of generality and of abstraction. In this manner, each structure prepares the way for the development or the acquisition of higher-level models.

The principle of unity

As one of its responsibilities, the teaching should cater for the cognitive, affective and social functions of the student. The process of interaction with the environment can be a rich source of affective and social experiences in addition to those of a cognitive nature.

The principle of activity

The learning process is the result of the student's activity. Of particular importance when considering the student-environment relationship are: the establishment of the habit of observation and development of the necessary skills required; the development of the ability to record information; and making use of experimental situations in which the student observes the results of his actions.

Recommendations that describe the student's role

Conditions should be created so that the student will:

Participate in the choices and decisions to be taken in the study process.

Possess the necessary data or criteria to enable him to decide if he has attained a predetermined goal or not.

Enjoy independence of action and the freedom to make attempts and commit mistakes on his own responsibility.

Be obliged to express in his own words (or by means of charts, schemas, or diagrams) the results of his work.

Learn to seek solutions on his own before asking questions.

Have opportunities for discussion.

Have opportunities for teaching his classmates or learning from them.

Define his own goals and learn to propose the means of achieving them.

Recommendations that describe the role of the teacher

He should produce a wide repertory of activities in which one or more mathematical models are related to social or cultural situations or to phenomena of the physical environment.

He should propose and accept proposals of goals, and help in clarifying and explaining them. He should aid the student in his choice. He should develop—with the student—criteria for determining whether or not the goal was attained. Once the goals are accepted, he should give his full support to the learning process.

He should encourage independent, creative and co-operative behaviour patterns. He should set store by initiative.

He should always be tolerant and attentive in his supportive role.

He should always be responsive to the positive aspects of the student's behaviour. Even the faultiest answer has some constructive aspect. His task is to discover this aspect and help the student to build upon it.

He should intervene only if asked to do so or if it is truly necessary (in both cases, his role should be one of action rather than words).

He should accept mistakes as a natural and inherent part of the research process.

He should not appear to be anxious to achieve results. The process is usually a slow one (what really matters is the fact that it is taking place, and that it is doing so in the learner's mind).

His attitude should give the impression of a teacher who is personally involved in what is going on and curious as to the outcome; the environment is a laboratory that allows everyone, without exception, to learn.

When his students are in doubt and cannot find an explanation for the phenomena they are observing, he should guide them by asking questions; he should improve his ability to ask questions, make suggestions and listen to what his students have to say.

When he asks a question, he should allow time for the answer; he should propose the basis for answering it, reformulating his question if necessary.

He should avoid empty questions that do not allow for an answer, since these accustom his students not to reply; in any case, a partial answer made by the learner is of greater value than a complete answer expressed by the teacher.

If he takes part in group work, he should adopt the tone and attitude of someone working along with the group; he should not impose his viewpoints; rather, he should make suggestions and allow the others to act.

If his reasonings are not accepted, this is because they are not suitable for the group or because they are not understood. It is useless—and contrary to the research process–for him to fall back on his authority. He should listen, look for another angle or for the flaws in his own reasoning.

Recommendations that indicate the way in which contents and their treatment are determined

Mathematics can exist as a process, or as structures of relationships. In the first case, stress is laid on the development of mathematical reasoning and in the second on the formal patterns that derive from it. Both forms should be present in the teaching (Dienes, 1971].

Learning mathematics is synonymous with constructing relationships. We can sharpen the image by saying that learning is transforming or building up structures, going from the parts to the whole, and from the whole to the parts. It involves integrating elements within a structure or transforming a structure so that it may incorporate a new element.

The factors which contribute to the construction of relationships are the search for regularities, classification, ordering, the development of models, the search for examples or counter-examples and the search for general terms.

The factors which contribute to the development of formal thinking are the description of phenomena, the effective use of reasoning, the use of symbols, the use of logical structure (if . . . therefore . . .), recourse to diagrams and charts and to verbal or arithmetical expression in the communication of thought and the use of symbols for communicating established ideas.

If concepts are to be active elements of the reasoning process, they must have 'content'. In other words, the student must be familiar with the elements that make up the whole which the concept defines. The concept may be considered to have 'content' if the student is able to give examples, to point out non-examples and to recognize the necessity for the concept in identifying the category to which the examples belong.

The teaching should provide opportunities for mathematizing actual situations. The resulting models should be described by the students at the outset in their own words. Once they display a relative mastery of the situation under study, they can proceed to do this in more formal terms.

Contents should be treated in context. Concepts and models should be inter-related with one another and with reality.

The mathematical language used by the teacher should be as precise as possible for the audience to accept it. The same precision cannot be expected of the participants (Davis, 1966).

Mathematical vocabulary should be kept to a minimum and transmitted through actual use. The use of formal definitions risks creating the illusion of a higher mathematical level. Often this only constitutes something superficial and is devoid of meaning for the learner.

Implications for teacher training

In the preceding section, a brief account was given of the characteristics of a methodology for the teaching of elementary mathematics. How is a teacher to be trained for this kind of approach? What elements, apart from the standard ones, should this training contain? Almost every aspect of teacher training requires some revision if strategies are to be used which stress the interaction of the student with the environment.

Mathematics training

In this area, it is preferable to select a few mathematical models and treat them in depth rather than provide general training. Rational and real numbers, Euclidean geometry and the use of algebraic algorithms should be included, as should the use of graphs, flow-charts and diagrams that illustrate relationships. It is useful for the teacher to be able to handle elementary physical notions, since mathematical models of velocity, force and electric current—to name a few—are related to everyday aspects of the environment. Generally speaking, the aim should be to familiarize teachers with the applications of mathematics, and, in particular, to provide experience in mathematizing everyday situations.

Training in method

In general, the teacher should be required to learn mathematics in a way in keeping with the methods which it is intended that he or she will apply. If, for example, he is to construct concepts with his students, he should first do this during his training. Among the topics to explore and develop, the following are important: the in-depth study of accumulated research on problem-solving, especially verbal problems; the teaching of applied mathematics; experiments in teaching laboratories; and experience in the development and further improvement of mathematical models based on situations.

The teacher as planner

Method which is oriented to the environment requires a teacher who is able to make decisions and to evaluate the results of his action (and not simply assess learning and achievement). As well as being given the opportunity to make decisions, he should be given the technical knowledge which will enable him or her to exercise this freedom. Special attention should be given to training in curriculum planning on the basis of information gained by diagnostic studies, planning strategies of learning, the selection and production of teaching aids, and the techniques and practice of formative evaluation of learning experiments.

The teacher as researcher

A main feature of the methodology stemming from the study concerns the student's ability to carry out research. For this reason, the teacher must have sound training in the techniques of observation. If only one recommendation were to be selected from all those made in this section, it would have to be this: the teacher must learn to observe, to record the results of his observation and to express them. It is this capacity that provides the basis for the skills that enable him to mathematize situations and fully understand the nature of the learning process (as distinct from that of teaching alone). As well as being trained to observe, the teacher must have some practical knowledge of techniques and procedures pertaining to the experimental method. These include those participatory techniques that have proved to be most suitable for communities or organized groups of people. The teacher must also be equipped with certain anthropological resources that will render him an effective observer of the social environment. Lastly, the teacher should receive a thorough training in the essentials of his stock in trade. The fundamentals of psychology, when accompanied by observation and the constructive evaluation of educational situations, can provide the teacher with a sound basis for interpreting and guiding the performance of his students. Moreover, educational philosophy, value-related analysis and the study of the main trends of contemporary thought can help the teacher in his role of guidance and in defining the conceptual context of his work.

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Gerhard Walther

Mathematical activity in an educational context: a guideline for primary mathematics teacher training

Introduction

In the past, many authors, (e.g. Hilary Shuard in her contribution to this volume) have pointed to the systematic character of education in general, of teacher education and of mathematics teaching, as well as to the 'open' variety of the variables that determine them, the interdisciplinarity of the variables and the consequential complexity of these systems. Nowadays we can draw upon a mass of empirical data and upon theories from pedagogical, psychological, sociological and other research on the learning and teaching of mathematics. And, out of this inflationary situation, there now flourish new programmes, research journals, special journals for the teacher of mathematics, and hundreds of books bearing upon the teacher's professional concerns.

How can one find an appropriate path in this jungle? And what does 'appropriate' mean? These questions are relevant and crucial for both the teacher or teacher-to-be and the teacher educator. For the primary teacher, who in general is responsible for subjects other than mathematics, the current scene is even more confusing.

One of the strategies chosen by large numbers of students to survive in this jungle is total adaptation to the system. This strategy, unfortunately, is not adopted because the theoretical knowledge is seen to be useful for later professional work, but because it is a means of passing examinations and getting a job. And there are deeper reasons. Theoretical knowledge is often thought to be harmful or obstructive to the teaching of children. So students in training rely more on their practical teaching experiences, their common sense and their own beliefs about teaching and education. They expect their mathematics courses to be mainly courses in method, where 'recipes' are given on 'how to introduce the addition and multiplication of numbers', etc. Such expectations reflect an inadequate grasp of the nature of education: the imitation of procedures is preferred to meaningful learning; an ability to explain 'how to do it' is substituted for meaningful teaching. Explanation is misunderstood. It is thought that one can reduce difficulties by 'atomizing' subject-matter into smaller and smaller portions of knowledge. On the other hand, teacher educators themselves tend to be fascinated by the practice of teaching. And this interest creates a danger of theoretical thinking being overwhelmed by practical action. This deep scepticism of theoretical knowledge is then transferred to the students and confirms them in their own attitudes. These difficulties are caused mainly by naive conceptions of theoretical knowledge and of the applications of theory.

J. Tamburrini (1975) discusses this failing in the context of Piaget's genetic epistemology and its implications for teaching. She writes:

> Piaget talks a great deal about the second misunderstood principle, that knowledge is a *construction* of external reality rather than a *copy* of reality. Again many teachers would insist that they do not base their practices on the empiricist notion that the mind of the learner is a clean slate on which the teacher can write. Nevertheless, if you go into the classrooms, you find a lot of practices that indicate that they really *do* have such an assumption . . . There is often the assumption, for example, that because you have provided a lot of interesting concrete materials, with the opportunity for the children to explore them at different times, that you are not succumbing to this 'clean slate' principle. But if you probe a little you will find that sometimes it is assumed that because in some way those materials embody a concept, then exposure of the children to those materials will mean that they automatically acquire the concept.

If pupils in the second grade are encouraged to make funny shapes with tiles that are congruent squares or triangles, it will be a nice game for them. But the new idea is the concept of invariance. This usually needs mediation by the teacher, who will ask the children to compare the number of tiles in different tesselations of the same shape. The concept will not, of course, be taught in a dogmatic way, but rather in the context of 'measurement'. Later on, the idea of 'ratio' must also be negotiated in the classroom. In this manner, the social aspect of knowledge—knowledge as shared knowledge—is taken seriously.

There is another misconception about applications of theory, which could be called the 'algorithmic view'. Here, theoretical concepts are interpreted as variables to be filled with situational data in order to obtain—in a strict, direct way—rules for action. This view neglects, among other epistemological facts, the transformation process from 'knowing that' into 'knowing how' (Gage, 1978; Skemp, 1979), a change that depends on the context of application and that, in general requires further development of the theory by the practitioner. In this sense, theoretical knowledge from the teacher's professional areas provides awareness, a means of orientation and possibilities for action. The hard job of applying theoretical knowledge in an exploratory situation has still to be done by the teacher. How can this latter mode of knowing be conveyed to the teacher in training? Recently, a remarkable attempt has been made in an international project called

'BACOMET' (Basic Components in the Education of Mathematics Teachers). BACOMET aims at identifying fundamental ideas for mathematics teacher education. By 'Basic Component' is meant an item that is:

Fundamental, in the sense that it plays a decisive part in the functioning of mathematics teachers.

Elementary, in the sense that it is accessible to intending teachers (so that it would be of immediate interest to those aiming to become teachers of mathematics and would introduce the student to, and prepare him for, important didactical and practical functions, both motivating him and enabling him to become acquainted with such functions).

Exemplary, in the sense that it exemplifies important didactical or practical functions of the teacher and their interrelationships.

Such a comprehensive, holistic conception of the ingredients of the education of teachers attempts to provide general orientations and awareness, as well as knowledge for action. It is in sharp contrast to 'methods' and 'recipe-oriented' teacher-training courses. The latter usually led to rigid and short-lived 'solutions' of the problems the teacher-to-be will later meet in the mathematics classroom. The former, it is hoped, will provide adaptive attitudes and strategies for innovative teaching and for further learning on the part of teachers.

Teacher education in mathematics depends heavily upon its cultural context, the educational traditions, the societal aims, etc. This is especially true for developing countries (see, for example, the illuminating case-studies of P. Gerdes (1981), relating to the situation in Mozambique, and of S. O. Ale (1981) concerning Nigeria). But, in all countries, teacher education in mathematics is subject to local conditions. In the Federal Republic of Germany, for example, many of the current practices derive their meaning from the intention to overcome the failures of the reform in primary mathematics teaching that began in 1968. But, as well as local needs, there are at least two other important factors that should not be ignored: the stimulating influence of neighbouring countries, and the on-going development of mathematical education as a science, linked with a professional orientation in teacher education (Otte, 1979). Nevertheless there are common points of interest, as the work of Ale (1981) shows. In the course of his study, he identified four problem areas: teacher problems, student problems, society problems and problems of resources. In the second category, it emerged that a number of students' difficulties are really caused by their teachers. Quite apart from 'poor teaching', the students indicated other subtle reasons, amongst them:

Constant discouragement. Some students complained that they had never succeeded in solving a mathematical problem of their own, except those already solved by the teacher, and these they often memorize.

Lack of problem-solving techniques.

Lack of skills and talents. The general belief of more than half of the students was that mathematics is only for the gifted. No matter how hard one tried, it was impossible to understand the subject, unless one was especially gifted.

The non-experimental nature of mathematics.

Too many incomprehensible formulae.

Excessive calculations.

Although these complaints come from secondary-school students, they indicate prevalent shortcomings of primary schools. What seems to be wrong, and this is expressed in the students' uneasiness, is a misconception on the teachers' side of mathematical knowledge, its transmission and its acquisition. One of our central aims in teacher education is to provide such general orientations. Before returning to this point in more detail, an example of practical work with teacher students in a primary class will be given to illuminate some of these ideas.

Teaching written multiplication in grade 3

The children were already familiar with multiplication by one-digit numbers. The aim now was to introduce the algorithm for written multiplication by two-digit and three-digit multipliers. We began with a problem closely related to the environmental studies recently made by the children. The problem was: how many hours are there in one year?

The students first encouraged the children to guess the number. This gave answers between 2400 and 10 000. Whose answer is nearest to the exact result? How can we find the exact number of hours? We must calculate 363 × 24. How could we do this calculation? No answer. Try to do it!

In the context of their previous work, this task obviously was a problem for the children because they had no straightforward algorithm at hand to use. Instead, they had to construct for themselves a tool to do the work, using their previous knowledge.

In traditional teaching, the teacher would have taught the multiplication algorithm by means of examples. He or she would have explained the rules, and, after a while, the children would have imitated the procedure to do similar tasks. But, would they, in so doing, ever come to grasp the meaning of this algorithm?

Let us look at what actually happened in the class. Almost all the children eventually got the right answer: 8760 hours. What, however, was interesting were the various ways in which they solved the problem. Essentially, there emerged five different types of solution:

Solution 1. Using addition only. The number of days in the year, 365, is written down 24 times and the total is found, namely 8760.

Solution 2. The number of hours in one day, namely 24, is decomposed into 10 + 10 + 4. The number 365 is multiplied, in turn, by 10, 10 and 4. The three products are added to arrive at the same correct answer.

Solution 3. In this solution, 24 is decomposed into 20 + 4. Now, 365 is multiplied by 20 and 4, and the two products are added.

Solution 4. The number of days in a year is decomposed into 300 + 60 + 5. Now 24 is multiplied, in turn, by 300, 60 and 5. The three products are added.

Solution 5. This involves double decomposition! 365 is decomposed into 300 + 60 + 5. Then 24 is decomposed into 20 + 4. Six products are calculated, namely 300 × 20, 60 × 20, 5 × 20, 300 × 4, 60 × 4 and 5 × 4. The six are added to give, once again, 8760.

At the end of the lesson, the student-teachers had a good feeling. The children were interested. They had worked mathematically in finding their own ways to the solution. But the regular teacher of the class did not share this enthusiasm. 'Where', she objected, 'was the straightforward introduction of the algorithm?' 'Why did you spend so much time letting the children use their "old" procedures? Wouldn't it have been better to use the time teaching the new algorithm to the children?' Indeed, the failure to introduce the new algorithm was fair criticism. But how could the lesson be brought to a good end? The students' proposal was to make use of the work the children had already done. This was generally agreed, and the aim of the next day's mathematics lesson was to answer two questions: In which ways did children actually calculate 365 × 24? How could the computation be simplified?

In the first part of this lesson, the children should discuss their solutions. They should explain their own procedures. They should find out how and why their different computations had worked. They should compare the calculations as to the time taken, the effort put in, their simplicity, etc. The teacher would stimulate and organize this discussion, but in a reserved manner. After these considerations, the standard algorithm would be introduced in the second part of the lesson as an abbreviated form of multiplication that would not be completely new, but would be close to the methods some children had used themselves. Children whose solution was more 'remote' from the algorithm would not be discredited, as they also had found the correct result, and their contribution had added to the interest of the lesson.

Although the children were not accustomed to this style of learning, they quickly adapted to it and became involved in the discussion. Some children, for instance, criticized 'complicated' solutions. Their comments included: 'You need not calculate 365 × 10 twice' (as in Solution 2); 'I can do it (365 × 20) quicker'; 'That's wrong, you did not multiply' (referring to Solution 1). In this, the teacher's role is to get the children talking about and reflecting upon their preceding activities. At this metalevel, the children should also learn that a mathematical task can be done in very different ways. These ways have been determined by the children themselves–not by the teachers or the textbook. Another experience, which the teacher should make explicit, is that each child can contribute to the common task, and children can learn from one another. The teacher has to mediate between individual knowledge (the different ways of finding a solution) and the common knowledge that is necessary for understanding the next mathematical procedure (multiplication algorithm). In this process, with the help of the teacher, relationships are established between the various modes of calculations (parts of knowledge) and the 'new' knowledge. So, the new algorithm grows out of knowledge that is shared. To return to the lesson, the discussion finally circled around Solution 3. This method was seen as the simplest one. Besides, the children also noticed and remembered that they 'have already done such multiplications'. 'Couldn't we combine both multiplications into a single one?' The student-teacher put the new problem. Initially, the second part of the question caused a lot of confusion. Eventually, he made explicit that he wanted 'to have only two lines under the multiplication bar (instead of three)'. Without further help, several children found the usual algorithm. Although this account can convey only a fragmentary impression of all that really happened, it is hoped that some features of mathematics teaching have emerged. Teachers need an adequate 'picture' of the nature of mathematics, especially of mathematical activity. In the conventional textbook (used in primary school), the task '365 × 24' is, at best, used to introduce the written algorithm in a direct way, or as an exercise to be performed after its introduction. But, in 'real' mathematics, it is a rare event to find for a new problem a ready-made method of solving it. This is also the case in daily life. A problem arises and one must master it in a more or less ingenious way, using one's own mental and objective tools. No one will have shown you before how to handle exactly that problem. In the lesson described, the student-teachers stimulated real mathematical activity with ordinary subject matter. The children were given the opportunity to indulge in divergent thinking, to discover ad hoc solutions, to make a break with routine procedures, to develop and/or apply heuristic strategies (e.g. the decomposition of the multiplier, reducing a multiplication task to a summation task, etc.), to communicate, to reflect and to argue about their activities. The teacher who aims at educating should trust in children's mathematical productivity. He should take their contributions seriously. He should conceive his role as that of a mediator between individual mathematical knowledge and the conventional mathematics he wants the children eventually to master. This role has been discussed, amongst others, by H. B. Griffiths (1975), whereas J. Brophy (1981) calls attention to the distorting effects that heads of schools, heads of mathematics departments, parents, teachers and pupils can exert on this conventional knowledge. The next section will discuss some important general guidelines for primary mathematics teacher education that subsume the aspects involved in our above example.

To teach mathematics means to do mathematics with children in an educational context

Children, as well as adults, invent, apply and learn mathematics in contexts other than educational ones. Here are some very personal examples. They are observations made of my five-year-old daughter, Sabine, and are typical of their kind.

Example 1. One day, whilst Mama was brushing her hair, Sabine took two almost-square shaped booklets. She crossed one over the other and cried: 'Look, that's a true star.'

Example 2. A few days later, when we were sitting at the table, she asked: 'When will the kindergarten festival be?' I answered: 'In four days'. She asked again: 'What day is to-day, Dad?' 'Thursday'. Then, after a moment of serious thinking, she replied: 'Oh, Monday, Monday, I'll wear my rope-dancer's mask.'

Example 3. Some days ago, when driving on the autobahn, she was bored. Once again, she played car-counting with my wife. She wanted to count the blue cars. My wife chose the white. After a while, Sabine suddenly took a sheet of paper. She cried: 'Let's do it again', and started to make two lists of strokes to represent the counted cars of each person.

Example 4. In a fortnight, she has to go to the hospital. Five days later we will be going to visit her Grandma. She asks: 'How many days to the hospital, Dad?' 'Fourteen.' She starts counting aloud from one to fourteen. 'And how many days to Grandma?' 'Nineteen.' Again she is counting from one to nineteen. 'Couldn't we go to Grandma before the hospital?'

In all these cases, Sabine did some intuitive, spontaneous and sometimes even sophisticated mathematics. But not in an explicit educational context as in school, where mathematical activity is used to achieve general educational goals. It is this particular intention of mathematics teaching and learning in school which distinguishes it from other contexts where mathematics is one, as in research or in mathematical applications. So, the teacher should be aware that, in most cases, school mathematics is a simulation of 'real' mathematics. Children, investigating, for example, a problem like 'How much does it cost to keep a dog?' (Tammadge, 1971) should learn, among other things, that mathematics can be used to model a situation, to process the data of the model, to predict something (e.g. the costs) and to make decisions on that basis. Though this very nice project seems to be taken directly from the children's environment, the teacher should be aware that the question posed is only one aspect of many different 'real' situations. The question could arise in a statistic context. Or it may be related to other issues such as giving to charity instead of keeping a dog, or to whether to keep a dachshund, a cat, or a love-bird, or simply to show little Tim that keeping a dog, besides being fun, also costs money. Someone who wants to know the cost of keeping a dog in his situation, and for his purpose, is hardly likely to imagine other situations or other purposes. To do so would be rather devious. In an educational context, however, the situational parameters and the parameters of the environmental context should be varied, so that students will appreciate the variety of issues that can arise.

There are also features of 'doing mathematics in an educational context' that have no equivalent in other mathematical contexts. One of these—and an important one—is the drill and practice of mathematical facts and routines. What, then, are the invariant elements of doing mathematics in school and doing mathematics outside school? H. Winter (1975) discussed this problem in the broad setting of general objectives for mathematics education, and, in doing so, laid open the deep relations between the anthropological, epistemological and societal/educational aspects of man and aspects of mathematics. As a consequence of this analysis, he postulated four conditions for meaningful mathematics education. The students should be given the possibility of being self-active, being involved in rational argument, experiencing the usefulness of mathematics and acquiring formal skills.

According to a suggestion of E. Wittmann (1981*b*), the last condition refers to learning facts, fundamental mathematical techniques and algorithms, whereas the others can be subsumed as cognitive strategies. For our purposes, it is important to describe the cognitive strategies in terms of student activity, for they constitute the invariant elements in the mediating process we were looking for. So, in 'being self-active', it is suggested that the student should work in an explorative, constructive way: making observations (of relations, patterns, structures); making guesses about observations; explaining observations and guesses; investigating special, 'illuminating' cases (in order to make general aspects transparent); making plans for solutions; developing reasons; organizing mathematical work (e.g. combinating partial solutions); generalizing; using analogies; going beyond given information and data; making variations of the 'given' situation; and generating related (or 'new') problems. When 'being involved in rational argument', it is suggested that the student learn to: discuss, compare and evaluate mathematical results; give examples and counter-examples; verify general propositions; and give reasons, proofs, etc. And, in 'experiencing the usefulness of mathematics', the students should learn to mathematize situations (inside and outside of mathematics). Thus, they should: describe and represent situations by mathematical means; collect data (by measuring, estimating, etc.); design mathematical models; process data; and interpret data, solutions, etc.

Mathematics has many facets. For primary-school mathematics, it is especially important to view mathematics as, in G. Polya's phrase, 'mathematics in the making'. So, doing mathematics with children in an educational context should be governed by the goal of developing cognitive strategies. There are at least three important consequences for teacher training. Teachers-to-be should be given the opportunity to do mathematics in the same spirit. They should reflect upon their activities and so learn how to initiate cognitive strategies in the mathematics classroom. In our example, we have already touched on several conditions for initiating cognitive strategies. A summary will be given here. Mathematical content should be developed by means of problems. These problems should be made accessible with the help of the teacher, who will take into account the children's cognitive conditions. Inciting activity by 'tasks' is a typical feature of the educational context. In the past, routine exercises dominated. Today, the 'task' is used more consciously, and is an object of educational research. From Pollak (1970) and from Avital and Shettleworth (1968) we are acquainted with the following types of non-routine tasks: open problems and challenging problems. With the former, the conditions of the problem are somewhat vague and not completely explicit. Such a problem can be pursued in various permissible ways. The necessary data and information must often be uncovered, as with 'How much does it cost to keep a dog?' For challenging problems, the available methods and knowledge only permit the problem to be understood and treated partially in a crude manner. Avital and Parness (1978) discussed another interesting type of problem, which they call an 'exploratory problem'. Though these categories are not mutually disjoint, the teacher should be aware of their educational intention. This is to break up the drill of routine exercises, and to bring some of the flavour of genuine mathematical activity into the mathematics classroom. Avital and Parness give seven criteria for exploratory problems. They are:

The problem must be amenable to an inductive investigation—that is, the student should be immediately convinced that the collection (and production) of data is the means to help him reach a solution, or a hypothesis.

The solution of the problem must appeal to the student as a goal for the attainment of which it is worthwhile to strive.

The collection of data itself must be of various levels of difficulty, so as to give the student a feeling of some accomplishment through the accumulation of more and more data.

Subgoals of gradually increasing difficulty must be formulated so that every child can contribute and obtain reinforcement at his level of ability.

While investigating the problem through the collection of data, the student be practising an important skill.

Some subgoals can be attained in a short time.

We may add one more requirement to ensure that the exploration will deal with a true problem and not with an isolated puzzle. This requirement is that the problem can be expanded to generate new goals.

Spread over the literature, there are many examples of exploratory problems. As illustrations, take the following:

Can the number '100' be written as a sum of consecutive integers (not necessarily beginning with 1)?

Take any two-digit number (a one-digit number can easily be made a two-digit number by the help of 0). Calculate the product of its digits. Repeat the process. What happens?

Are the tall children in the class also the heavy ones?

Draw a convex polygon. Triangulate it. Are there relationships between the numbers of vertices, diagonals and triangles?

The seventh criterion of an exploratory problem was that it could be expanded, to generate new, related questions and to open up new roads of investigation. Appropriate methods have been suggested by Brown and Walter (1970), by Wittmann (1971) and by Polya (1966). The teacher, furthermore, should incite divergent thinking, cognitive conflict, heuristic procedures, intuitive arguments, discussion and a constructive attitude towards errors. A simple example related to heuristic procedures in the context of the hours-in-a-year problem arose when the student-teacher asked students who had already done their work to compute the number of hours in a leap-year. Some of them started their computations completely anew. Others used the results of the previous problem, an important heuristic method which could afterwards be made explicit in classroom discussion.

How can future teachers be prepared to do mathematics with their children at school?

About twenty years ago, Polya (1963), in his 'On Learning, Teaching, and Learning Teaching', seeing 'teaching as an art', developed perspectives on teacher training in which the acquisition of the necessary knowledge for action for the teacher-to-be is mainly left to him. In addition to the official recommendations of the Mathematical Association of America, Polya proposed that:

The training of teachers of mathematics should offer experience in independent, creative work, at an appropriate level, through a problem-solving seminar or through any other suitable medium.

Methods courses should be offered only in close connection either with subject-matter courses or with practice teaching, and, if feasible, should be given only by instructors experienced both in mathematical research and in teaching.

For primary-school mathematics teacher education, these proposals perhaps seem to be too demanding and pretentious. On the other hand, primary-school teacher education, in my opinion, should, at least, be guided by the spirit of these maxims. That means that the teacher should do mathematics, and should reflect such activities in the educational context. As to Polya's second demand, Freudenthal recommends a more practible method which accords better with the needs of primary-school teacher training:

Change in teacher training at the primary level requires a far-reaching integration of teaching subject matter with its didactics–and as a precondition for the last: that the subject matter by the future teacher is so close to the subject matter to be taught by him in his future career that it is accessible to this complete integration with didactics.

Again Polya's pragmatic conception of teacher training is expressed in vivid words by Halmos (1975):

The best way to learn is to do, to ask, and to do.
The best way to teach is to make students ask, and do.
Don't preach facts–stimulate acts.
The best way to teach teachers is to make them ask and do what they in turn will make their students ask and do.

Apart from several determined efforts at regional levels, Polya's ideas on teacher training are still far from being realized on a broad basis. At first sight, this seems to be in constrast to the huge amount of problem-solving literature produced in the last twenty-five years. If we look, however, at, for example, the 1976 report of the ICME Study Group *Problem Solving, Teaching Strategies and Conceptual Development,* or the article by Krulik and Reys (1980) in the NCTM yearbook on 'Problem Solving in School Mathematics', it can be seen that the centre of research and interest is 'Problem Solving in the Classroom'. In the light of this goal, studies in training are sold short. It will not do–and I would like to repeat and stress this again–to expose students only to theories on problem-oriented teaching (its conditions, its character, its possibilities, its difficulties, etc.). What must be done is to provide such training that problem-orientation and 'doing mathematics with children' will become fundamental to the role of the teacher–a part of the teacher's personality and an essential element in his professional life.

In the Institute for Mathematics Teaching, Kiel, the students begin their training with a two-semester introductory course. In this course, we begin with Polya's classical problem: 'Into how many pieces is space cut by five planes?' I choose this problem because I can be relatively sure that none of the 200 beginners has ever met it. So the experience of wrestling with it will be a new one. There is another reason. We have at our disposal Polya's film *Let's Teach Guessing*. Here Polya works with a group of mathematics students as they tackle this particular problem. So the film offers the students an opportunity to see some of their activities and errors 'mirrored' after making their own efforts. It also gives me the opportunity to introduce Polya, some ideas about heuristics (which can now be illustrated by the students' own work), some ideas on Polya's concept of learning, teaching and teacher education and some general ideas on the development of knowledge linked with the names of Piaget, Popper and Lakatos. At the end of this unit, the students have to study a brief article of Polya on teaching problem-solving. This is then discussed in smaller groups of thirty to fifty students in the light of the students' experiences of their own school days.

Most students seem to be helpless when they confront the problem. Some, perhaps, are searching for a formula, since others explicitly remark upon the lack of one. Some students trivialize the problem by assuming the planes to be parallel. So the problem itself has to be laid bare. What assumptions can be made? When we have agreed on the general position of the planes, how can we find the number of regions? Obviously the method of guessing or hypothesizing has been blocked by a long period of school mathematics, so it must be introduced anew. Many students feel embarrassed if they say something wrong. But when the law 2^k for the number of space regions, if k is the number of planes, seems to come out, the students are suddenly encouraged to predict the number of regions, even for nine or more planes. Guesses should always be tested. Systematic counting is necessary for three and more planes. What a disappointment when it turns out that four planes cut space into only fifteen regions! Without systematic counting, most students do not believe that number. The good-looking hypothesis that 'each new plane divides each previous region into two' has turned out to be wrong.

The 'into two' guess obviously results from assimilating the space-cutting 'halving'—a schema which, when confronted with reality, caused a conflict. The conflict can only be resolved by accommodating the halving schema to reality. It is no error to hang on to the halving schema because it works for zero to three planes and in fact describes what is going on. Ginsburg's comments on childrens' mistakes comes in mind. They:

are often organized, and rule-governed, and have sensible origins. It does not seem helpful to characterize mistakes as capricious, and it is not helpful to attribute them too much to low intelligence, learning disabilities, or the like.

To resolve the conflict in our problem, one could investigate analogous problems: 'Into how many pieces is the plane (the line) cut by five lines (points)?' There are also combinatorial routine procedures to solve the problem, which are not, of course, at the students' disposal. We use a 'situative' solution that follows quite naturally from the students' halving-hypothesis. This procedure also sheds light on an important teaching and learning method.

To accommodate the halving schema, it could be asked: What happens to the number of regions of space if we add to the present three planes a fourth plane in a general position? The three planes could be visualized as a tent, or a space-corner in the room. We already know that the three planes divide space into eight regions. The fourth plane can be visualized as the floor of the tent. This plane does not halve each of the eight regions. It halves only those regions which are cut by the fourth plane. These regions can be counted by means of the regions in which the additional plane is divided by the three original planes, or by the three lines of intersection with these planes. Three lines generally cut a plane into seven regions. So, for four planes, we have $8 + 7 = 15$ regions of space (instead of $2 \times 8 = 16$).

Until now, we have worked with a single example, where we could confirm and explain the wanted number, 15. Reflecting upon the operation and its effects, linked with our question 'What happens with . . . if . . .?', it can be said that we have a procedure that applies not only in our special case, with three planes initially, but also in all other cases. That means that the general case has been illuminated by a representative example (Walther 1979, Semadeni 1981). Such an example provides a strategy for arguing the general case. Halmos reports that Hilbert once said:

> the best way to understand a theory is to find, and then to study, a prototypal concrete example of that theory, a root example that illustrates everything that can happen.

Abstraction does not generally result from the objects alone, but, essentially, from invariants connected with operations acting upon the objects. According to Piaget this latter kind of abstraction, which plays an important role in mathematics, is called *reflective abstraction* (c.f. Wittmann, 1981*a*.).

The above question, 'What happens with . . . if . . .?', is intrinsically linked with the 'Operative Principle' of learning and teaching. This initially was based on fundamental issues of Piaget's theory, embedded later on in a broad epistemological context of 'Operative Programmes' (Wittmann, 1981*a*.).

The advantage of using representative examples in teaching and learning is obvious. The representative examples may be material objects such as buttons, Cuisenaire rods, tiles, mirrors, etc. But they must be generic. That is to say, not too simple, not too complicated and without extreme features. Working within these limits makes it possible to avoid the formal machinery necessary to represent the abstraction (in our case, a recurrence relation). But doing so yields structural insight and understanding. So the formal representation need not be envisaged until the meaning of the situation is grasped. The use of representative examples has a long tradition. Euclid's famous theorem: 'Prime numbers are more than any assigned multitude of prime numbers' (Book IX, Proposition 20) is proved for a set of three prime numbers a, b, c. Either $(a \times b \times c + 1)$ is a prime number different from a, b or c or $(a \times b \times c + 1)$ has a prime divisor different from a, b or c.

Simon Stevin in his work *La Disme* always used representative examples to discuss general facts from the theory of decimal fractions. It is interesting to analyse his method. In the 'Proposition' he stated a problem: 'Given three decimal numbers, 27.847, 37.675 and 875.782 (in our notation) it is required to find their sum.'

In the 'Construction', he described the procedure used to add these decimal fractions: 'Arrange the numbers as in the accompanying figure, adding them in the usual manner of adding integers . . .' The 'figure' showed the usual vertical setting which we teach to children, with the units, tenths, etc. digits in the same columns. The 'proof' was given by translating the numbers into vulgar fractions and adding them according to the usual rules.

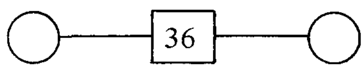
Two examples from primary-school mathematics follow. First, by representing the product of two natural numbers m and n as the area of a rectangle, the law of commutativity can be illuminated. For example, the product of 5 and 3 can be shown as the area of a card 5 units long and 3 units wide. The turning of the card through one right angle 'shows' that $5 \times 3 = 3 \times 5$. An explicit formulation of the proof by rotation is unnecessary. Basic experiences of rotation show that the area of the rectangle remains invariant, however it may be orientated.

Another example from primary-school mathematics can be given. The relation $(m+n) = (m+t) + (n-t)$; where $t \leqslant n$, and m, n and $t \epsilon N$, (which is important for simplifying the addition of natural numbers) can be illustrated with a representative example: $(8+3) = (8+2) + (3-2)$ in the following way: A rod of length 8 is put into alignment with a rod of length 3. It can then be 'seen' that $(8+2)$ and $(3-2)$ are the same as $(8+3)$.

The use of representative examples makes it possible to give a formal proof in a more conscious and sensible way. As a way of modelling a general statement, it provides us with a powerful tool for enlarging the range of applications of mathematics.

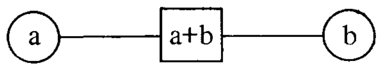
Let us now return to our initial problem, that of dividing space by planes, and consider its further role in our teacher education course. In various places in this article, I have deplored the division of educational knowledge into separate compartments, labelled content, methodology, psychology, epistemology, educational theory, etc. I am aware that specialization in the various sciences relevant to teaching and to teacher education is responsible for this division of labour, which is then imposed upon teacher education. Nor am I against specialist courses, because nobody can be an expert in all these relevant fields, but it is usually left to the student himself to integrate the separate parts and aspects of educational knowledge into a comprehensive whole. To balance this situation, teacher-training courses should contain integrating units. Our above problem is such a unit. And, in training for mathematics teaching, it is appropriate to choose a mathematical topic (a problem) as the nucleus of such an integrating unit. Freudenthal recommends choosing problems from primary-school mathematics that are rich and elastic enough to serve also as starters for students in training. As an illustration, let us consider the integrating unit 'Arithmo-chains'. McIntosh and Quadling (1975) invented 'Arithmogons'. These are the closed form of arithmo-chains. Work with children showed that chains are easier to survey, and hence more appropriate than arithmogons.

Short arithmo-chains are of the following type:



5	31
12	24

Here, the input number 5 yields the output number 31. And the input number 12 yields the output number 24. So, the general rule of construction is:



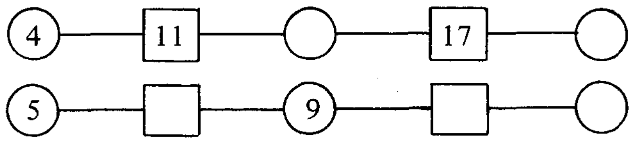
With such chains, various issues arise:

Can the output number equal the input number?

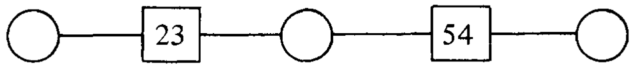
What happens to the output number if you increase/decrease an input number by 2, 3, . . .?

Show what is happening with chips or graphically. What happens to input/output numbers if the number in the rectangle is changed (increased/decreased, multiplied/divided)? Guess relations.

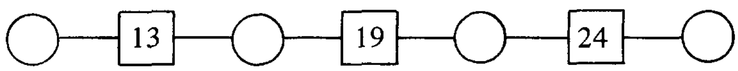
Now, short chains are connected:



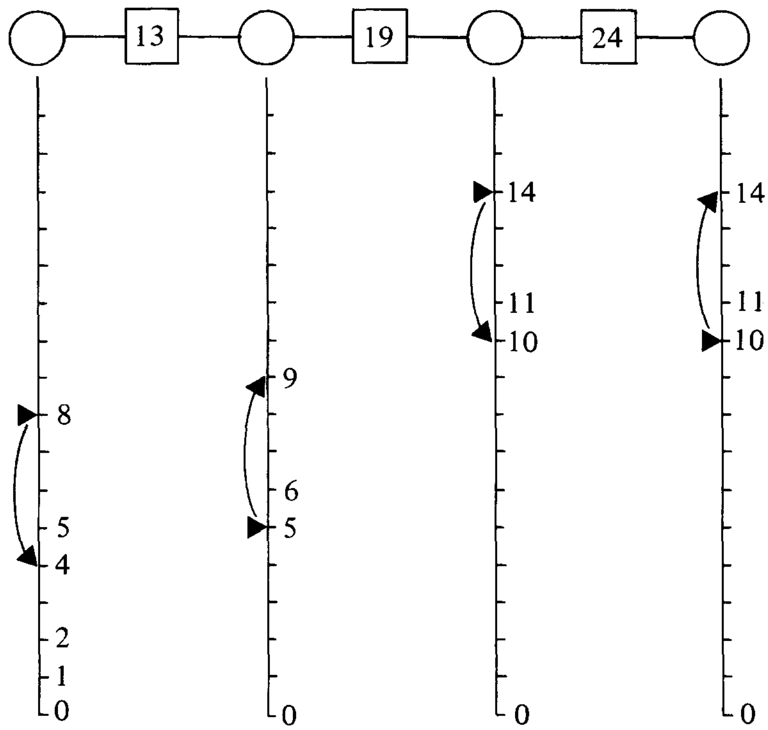
Fill the blanks according to the rule.
Can you put three numbers in other positions (blanks), so that the two missing numbers are uniquely determined?
How many different cases?
Construct other two-chains with three or fewer numbers, so that at least one of the missing numbers is not uniquely determined.
From now on we consider chains of the type where all numbers in the rectangles are given:



Which output number belongs to the input number 5?
Can the output number equal the input number?
Some of you certainly will use equations, but children in grades 3 and 4, do not, in general, have this technique at their disposal.
Try to explore the situation in a quasi-empirical way.
Choose several input numbers.
What happens to the numbers in the other circles?
Make a table.
Is there a pattern?
What happens to the output number if an input number is increased/ decreased by a certain amount?
Is the pattern related to the given data? (Here, the difference between the output and the input number equals the positive difference between the numbers in the rectangles.)
Try to give reasons for your guesses.
Call an arithmo-chain solvable if there is an input number that equals its output number.
Are two-arithmo-chains solvable?
Try to characterize solvable two-arithmo-chains.
How can one solve three-arithmo-chains, such as:



In the classroom, I use a magnetic blackboard to visualize the relations between the several numbers involved, like this:



The vertical lines are graduated into 16 or 17 equal steps, with zero, 1, 2, 3 etc. all at the same horizontal level.

With the help of coloured markers, the respective numbers in the circles can be shown on the vertical number lines. This device makes it possible to study the dynamics of the relations between the numbers (especially if each of four children is responsible for moving the markers on the number line).

I ask: 'What do you observe watching input/output numbers?' Sometimes a hint is necessary here, which I give in the form of a related task: 'John has four marbles, Mary has fourteen; how many marbles must Mary give to John so that they both have an equal number?' This situation incorporates the idea of the arithmetic mean, which finally yields a solution of our arithmo-chain.

Can you choose a simple input number so that the output number almost yields the solution of the above three-chain? (Take 0 or 13).

For teacher students: How could you show by means of representative example that every three-chain is solvable?

Describe your strategy with variables.

Comment on procedure with primary classes.

We intentionally do not restrict the domain of numbers. So frequently pupils choose for, say, the above chain, an input like 15. What now is the number in the second circle? In our syllabus, negative numbers are officially introduced only in grade eight. On the other hand, here is a good occasion to introduce negative numbers informally, via the permanence principle (Freudenthal calls it the 'inductive-explorative method').

The next step is to investigate four-chains. Now the students work like children at school. Hence, from their own difficulties, they get a feeling for the children's difficulties. But they do a little more. Certain aspects are made explicit and are discussed. These are generality, the characteristics of solvability, the use of variables, arithmo-chains as functions with certain properties, work according to the operative principle, use of teaching aids, etc. The mathematical level can be further raised when the students have become familiar with the behaviour of arithmo-chains of specific lengths. They then guess and prove a general statement about the solvability of n-chains.

There is another path for students to investigate. Students from former courses and I have tried out this topic several times in primary classes, and we have made several video recordings. These video tapes can be analysed with new students. They can identify teacher initiatives, pupils' initiatives and various pedagogical categories. So, using this integrating unit, students are introduced to lesson-planning, to developing exercises, to materials, to work cards (or to a textbook page), to analysing which specific arithmetical skills are involved in the topic, etc. It is agreed that some, at least, of the students will eventually try out their work in the classroom, and that the others will contribute in analysing the video recordings. The topic can also be used to introduce students to 'clinical' investigations with a particular child. Here they learn to observe carefully how a child works, to intervene if necessary, and to analyse and explain the processes involved. Such work, of which many examples can be found in *The Journal of Children's Mathematical Behavior,* will contribute to making students open-minded towards the human dimension of teaching and learning.

Conclusion

The course, which we call 'Introduction to Mathematics Education', is not entirely filled by the themes and aspects described above. Others are interwoven with them. They include: mathematical modelling (including aspects of 'word problems'), the role and diagnosis of errors (attempting theoretical explanations of certain types of errors), rudiments of the psychology of mathematics learning, texts in mathematical learning and teaching, the role of concepts, learning/teaching processes (e.g. communication patterns), etc. Several textbooks are available, though they have to be adapted in several respects to the special needs of primary teachers. They include: Farell and Farmer

(1980), Wain and Woodrow (1980), and Wittmann (1981*b*). These other themes are not, of course, discussed in detail. But they contribute to the general aim of the course, which is to germinate a fundamental orientation towards mathematics education, and to help students to construct their own 'interpretive lenses'. This attempt is confirmed by a recent article from Kilpatrick (1981), where reasons are given for 'the reasonable ineffectiveness of research in mathematics education'. His suggestions, amongst others, are to take theory seriously, and to view teachers as participants in research. They apply analogously to teacher training, if research and university teaching are seen to form an insoluble unit.

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David Johnson

Informatics: implications of calculators and computers for primary-school mathematics

The ever-increasing availability of technology in schools and in everyday life places considerable pressure on the teacher and on the curriculum. It is also the case that, with today's micro-electronics technology, mathematics is likely to be most affected. This is not meant to suggest that even greater impacts on the teaching of other subjects will not be made by such developments as 'easy to use' and 'inexpensive' word processors and computerized library information retrieval systems. However, at this point in time, calculators and micro-computers are already being used in many schools in the United States, the United Kingdom and other developed countries, and one of their major uses is to supplement and/or support the teaching and learning of mathematics. The availability of this technology raises many questions for the curriculum. For example: What mathematical skills and understandings are 'basic' for functioning in a technological society? Are there branches of or topics in mathematics that now take on more significance?

There are also very important questions about the role of these devices in instruction. In particular, the question: Where should the emphasis be placed—on their role as a tool or as a tutor? While this question may be more relevant to a discussion of computers, it also has implications for assessing the place of some calculators.

Calculators and computers: tools or tutors?

It is appropriate at this point to establish the overall aims and the context of this chapter. One could review, with minimal comment, the range of potential uses of calculators and computers in primary-school mathematics. This would include their tutorial function or their capability of giving drill and practice, their use in simulation and games and as devices for generating data for demonstrating and/or reinforcing concepts and, finally, their value in 'algorithm design' when children develop their own procedures or computer programs for processing by the machine. This set of activities illustrates the range or continuum from 'tutor' to 'tool'. To provide an adequate discussion of all these uses would require a book. Hence, for the purposes of this chapter, the view taken is that the greatest potential of the technology of calculators and computers is that of a tool for exploring mathematical ideas and for expanding the range of possible activities and topics which might be studied. This view embraces the idea that the technology is controlled by the user, that is to say, by the child, as opposed to the user being controlled or directed by the technology.

Before continuing, it should be said that there is a paradox in the widespread use of calculators and computers to check answers to long computations done with paper and pencil, or in using computers to present, for example, two-digit multiplication exercises and, subsequently, to provide feedback of 'correct' or 'wrong, try again' to the user's response. In such cases, it is quite clear that the teacher or school has decided that a high facility to perform the computational algorithms for addition, subtraction, multiplication and division is important and necessary, and, further, that to use computer technology in this way can enhance, or promote, the acquisition of these skills.

In general terms, the results of research support the use of the technology in skill-learning in mathematics (Burns and Bozeman, 1981). But this begs the real question: Should we be using the machine to train children to do tasks that are better done by the machine in the first place? Why did the computational algorithms or techniques first originate? Quite likely because there came a point when it became too demanding or difficult to recall all the 'basic facts' from memory. A mathematician friend of mine once noted that all sums and products are really 'basic facts'. We invented the algorithms for working with multi-digit problems only to save memorizing the results we might need. For example, the product '45 × 27' is really a basic fact. It can be written as '45 × 27' or more meaningfully as '1,215' (although in some contexts the first form, written as a product, may, in fact, be more meaningful and the more appropriate answer to a problem). Of course, the number of such facts that one might need, even in ancient times, was unpredictable. So it made sense to invent methods to obtain them reasonably quickly and accurately. (The development of the base ten numeration system and the corresponding algorithms for computation were really a quite remarkable achievement.) The mastery of these computational algorithms has traditionally been deemed very important, both for using mathematics in everyday life, and as a prelude to the study of additional mathematics. But are they really that important today?

Most teachers would accept the view that children need to be given the opportunity to explore mathematical ideas and to 'play' with mathematics (a philosophy inherent in many of the attempts to develop primary-school mathematics in the United Kingdom during the 1960s and 1970s). The world-renowned mathematician and mathematics educator, George Polya, Professor Emeritus of Stanford University, makes the point that mathematical thinking is not purely formal. It is not concerned only with axioms, definitions and strict proof, but with many other things. These include generalizing from observed cases, inductive arguments, arguments from analogy and recognizing a mathematical concept in or extracting it from a concrete situation. Polya (1965) goes on to make a plea that we should 'teach guessing'. His writings on problem-solving all reflect a concern for *doing* mathematics. However, it is often the case that playing with mathematical ideas gets bogged down with tedious arithmetic, and we find children have difficulty in doing the arithmetic successfully. One attempt to overcome this difficulty has been to spend a considerable amount of time in drilling the algorithms of arithmetic. To return to my earlier point, is this really necessary? It may be useful to understand how a numeration system works in base 10, or in another base. Early work with concrete embodiments of numbers (multi-base blocks, for example) will quite likely provide this understanding, and will also do much to enhance the understanding of place value. But, is the rote learning of algorithms for multi-digit computation through repeated practice worth all the time now being spent? I say no. The calculator can be used to recall the information needed quickly and accurately. When he or she confronts a problem, the child must still make decisions on what information or data are to be used, or processed. And to solve the problem, the child must also interpret the output. These decisions are the critical ones in processing information. Those who regard these views as 'heresy' often ask: but what if the calculator breaks down? Or what if a calculator is not available? My reply to this is: one goes and gets another, or waits until a machine is available, particularly if one is concerned for accuracy. And, if accuracy is not a concern, then one applies certain skills of estimation.

I should make the point here that I do not expect all readers to agree with the view espoused above, nor do I consider agreement to be desirable. My concern is to urge teachers and others to reflect on what is meant by 'basic numeracy' in a technological age. I will review what I consider to be basic numeracy later. However, before dealing with the specific questions of the classroom, let me briefly trace the history of calculating and computing devices. This will show that, as computation has become more essential to trade and commerce, man has sought to mechanize the labour involved.

Calculators and computers-a brief History

The British Science Museum has a fascinating exhibition that includes developments in mathematics on one side of a large room, and calculators and computers on the other side. Among the latter, there are a number of items that illustrate the continual reduction in the length of the time intervals between important consecutive contributions.

In considering calculating devices, one usually begins with the abacus–the use of beads on rods to assist with tedious calculations. A placard on the wall of the Science Museum indicates that the oldest known example of the use of this device is attributed to the Greeks a few centuries B.C. Other available information indicates that a particular form of the abacus was in common use as a calculating device in China until the thirteenth century, and that a modification of the Chinese design became established in Japan about 1600. This device, which is referred to as the 'Japanese abacus', has had wide use up to the present day. Thus it appears that, for more than 2,000 years, people have been concerned to design some sort of device for handling tedious, and often boring, calculations. So it is a peculiar feature of today's education that we seem to be moving in the opposite direction with our heavy emphasis on the ability to perform calculations rapidly and accurately with pencil and paper.

Another interesting development in the area of calculation was the design of the slide rule. Both the straight and circular forms were invented by an English country rector, William Oughtred, about 1621. However, the slide rule was dependent upon two earlier contributions in mathematics. The first, the invention of logarithms (the representation of any number as an exponent to the base 10 or to the base *e*), is attributed to the famous mathematician John Napier, Baron of Merchisten, near Edinburgh, Scotland. Napier published his book on logarithms in 1614. Second, it was necessary to develop the logarithmically divided line before one could apply Napier's invention to the slide rule. The logarithmically divided line was the contribution of Edmund Gunter, a professor at Gresham College. It was published in 1623. The slide rule has played a significant role in the development of science for well over 200 years. But now it has been quite definitely replaced by the scientific calculator and has become a part of history.

How and when did calculating machines come in? Actually, their history goes back to the early seventeenth century, the same era as the slide rule. Among the early designers were some mathematicians such as Leibniz and Pascal. However, no really reliable machine resulted from the many early, ingenious attempts. The first reasonably reliable

1. The history has been taken, with some modification, from Professor Johnson's Inaugural Lecture, 'Figures and Chips', Chelsea College, University of London, 20 January 1979.

calculator did not appear in the United Kingdom until the nineteenth century–the Thomas Arithmometer. However, from this point onwards developments came rapidly. In 1872, the barrel machine was invented. It proved to be the most popular type of general purpose calculator from about 1890 to the 1930s. In 1922 we had the first fully automatic, but still mechanical, multiplication and division machine–the Monroe Full Automatic Calculating Machine. Printing calculators were introduced about 1900, but they didn't come into real use until about the 1930s and 1940s. The first electronic calculator, the Anita, was developed and manufactured in the United Kingdom as recently as 1961, and by the early 1970s this gave way to a keyboard of ten numbers, digits and memory keys. 'Chip technology' had a major impact on calculators in about 1974 with the advent of HP65, the world's first programmable pocket calculator. The chip technology has had, of course, another major impact in that calculators have so decreased in cost that a simple four-function machine can now be purchased for under $10.

The development of computers closely parallels recent developments of calculators. Mechanical punch card, or automatic control machines, were used as early even as the late nineteenth century. However, while these were generally considered to be the first operational 'computers', one usually attributes the first developments to Charles Babbage (1791–1871) in the United Kingdom. Owing to the number of errors Babbage found in mathematical tables, he decided to build an engine that would calculate and print tables automatically. With the support of the British government, he began work on his first 'Difference Engine' in 1823. Construction stopped in 1833 when he moved towards developing a new general-purpose calculating engine (the name 'Analytical Engine' was first used in 1841). Unfortunately for Babbage, his design went beyond the practical limits of the manufacturing processes of the time and no working Analytical Engine was ever built. However, his first machines stand in the Science Museum and are quite impressive.

The first real breakthrough in the development of computers occurred in the 1930s and 1940s, some long time after Babbage. This brought the first electro-mechanical machine, the Harvard Mark I, which was developed and used in the United States between 1937 and 1944. The first general-purpose electronic computer, ENIAC, developed by Echert and Mauchly at the University of Pennsylvania, was actually running in 1942, although the working dates are generally given as 1946 to 1955. ENIAC was 1,500 times as fast as the earlier Harvard Mark I. From this point on, developments occur very rapidly, and it is difficult to establish an accurate time-line, or to give proper credit to individuals. Some selected contributions which eventually led to what we have today include:

The concept of a stored program. The first prototype to use this concept, and which actually ran successfully, was developed at Manchester, United Kingdom, in 1948.

The development of the transistor to supersede valves or vacuum tubes by Bell Telephone Laboratories in 1948. But it took until the late 1950s before machines were built that took full advantage of this technology.

Integrated circuits in the early 1960s.

The chip, capable of carrying thousands of components. By 1971, we actually had the entire central processor on a chip. This development provides the basis for the machines of the 1980s. We now have small desk-top computers that are faster, more reliable and considerably cheaper than those of the 1940s and 1950s. It is startling to think that ENIAC occupied 3,000 cubic feet and weighed nearly 30 tons, and that, whereas the early machines cost hundreds of thousands of dollars, we now have better equipment costing less than $500.

What does all this mean? Have we reached a plateau of development? The answer, it seems, is no. But it may well be that the next real development will be to produce home computers with the power of the now 'bigger' machines, along with new refinements in the applications and uses of this equipment. We see some evidence of this trend in that small, inexpensive home or school micro-computers are now available with colour, high-resolution graphics, substantial memory, etc.

We also now have 'computer languages' that are simple to learn and use, and soon the ordinary citizen will be able to control and operate his or her own computer. The developments in computer languages are as impressive as the developments in hardware. From machine coding in 'binary' or 'octal', we moved to what is called assembly language (alpha-numeric coding), and then to more general purpose and 'natural' languages, such as 'Fortran', 'Algol', 'PL-1', 'Cobol', etc. These are often referred to as 'high-level' languages. But these high-level languages have been refined, and new languages have been developed that enable almost anyone, including young children, to program the machine and explore ideas and relationships. The most common language available on most of today's micro-computers is 'Basic' (a language developed for the non-specialist). There are, however, a number of alternatives to this language, and, at the time of writing, two of them warrant special attention. One is 'Logo', and of particular interest is the work done by Seymour Papert with young children in the United States using the Logo Turtle to explore mathematical relationships using powerful graphics. Another new language, somewhat related to Basic, is called 'Comal'. This language has certain features related to its structure that make it particularly appropriate for general-purpose use. The key feature of many of the newer languages, however, is that they are not difficult to use.

Where does this bring us? These developments in calculators and computers were motivated by many, often conflicting, purposes. It is clear, however, that the main purpose of this technology was to enhance and to extend the mental capability of the individual. Man is now able to investigate and explore ideas and phenomena that previously were never possible within the whole lifetime of an individual.

There is also another important consideration: the scale and ever-increasing frequency of new technological developments. In this context, think of the 7- to 10-year old child in today's classroom. What skills and knowledge will this child need to continue in school, and, more important, when he or she enters the world of work and leisure? The changing nature of life in a technological society in such that education as 'learning how to learn' is of particular relevance. School mathematics offers many opportunities to explore and to investigate relationships that have been extracted from the child's environment. The next section of this chapter will attempt to make this idea more concrete and feasible for the primary-school classroom.

The real world of the classroom

What is meant by the phrase 'learn how to learn'? It may well be like problem-solving. George Polya says you learn how to do it by doing it. But this implies providing the opportunity within a context that is motivating to the child. I like to think of mathematics as a subject wealthy in opportunity and one where the nature of the subject provides the motivation to explore, and hence to promote 'inquisitive behaviour'. The question 'what do I see in this graph?', or 'what relationships do I observe in this set of numbers?', or, more generally, 'what happens if . . . ?' are all natural issues in the mathematics curriculum at all levels.

Where does the teacher begin? The answer is 'try something'. Before looking at some specific examples of things that might be tried, let me return to the issue of basic numeracy. As indicated previously, I have my own list (and others have theirs). This includes: facility with single-digit computation; ability to work with powers of 10; understanding of place value; and a number sense, which includes what to do when.

My arguments for including these four areas are really quite straightforward. First, one must be able to estimate if one is to assess the mathematical reasonableness of a result given by a calculator. A quick means of estimating is to work mentally with 'easy numbers'. I have published a paper (Johnson, 1979) that discusses this topic and offers suggestions for the classroom. Basically, the technique is to substitute for the given numbers other numbers whose form is that of a single digit multiplied by a power of 10. For example, '620 × 46' becomes '600 × 40', or '6 × 100 × 4 × 10'. Here it should be noted that the children need at first only to use the leading digit. Hence '46' becomes '40'. The computation of the estimate then becomes a task that involves working with single-digit numbers and with powers of 10. In order, of course, to understand how to perform this simpler computation, the child should have a feeling for the commutative and associative properties of numbers. It is also necessary to understand place value, particularly when the numbers involve decimals or when the computation includes, say, adding numbers that differ by a factor of 10 or more.

Nothing has been said about how best to teach these four components and I am not sure we have an adequate answer to this question. We certainly cannot say that children in school today have achieved a mastery of even the first three of the items on this short list of concepts or ideas. However, it is also accepted that these ideas are among those most difficult to teach.

The fourth item on my list, number sense, really includes a number of concepts. First, the child needs to be able to determine when to use one or more of the four basic operations and how to go about tackling a given problem. Second, I believe that children should be able to use numbers with ease and be able to observe and make use of mathematical relationships. I think this is an area that is often slighted in today's mathematics teaching. However, many of the books, now on the market, of number puzzles to be solved with a calculator include what I think are quite fascinating activities for developing 'number sense' as I understand it. Some examples follow. The feature of them is their missing digits or their missing operations. They can of course be done without a calculator, but it will be seen that a calculator would help, and would also make the activity more interesting or fun.

Missing Digits (each box can have any digit 0–9)	93 X 8 □ = 7 □□8
	□□6 X 84 □ = 232668
	3 □□4 ÷ 8 □ = 48
Missing Operations (each circle can have any operation +, −, X, or ÷)	(37 ○ 21) ○ 223 = 1000
	27 ○ (36 ○ 11) = 675
	619 ○ 316 ○ 425 ○ 196 = 924

It will be seen that these problems lend themselves to seeking out or to observing certain relationships. And there is a real advantage in using a calculator. It is that you can attack such a problem in a number of ways, including that of trial and error. For some children, this method may be their only way of getting started. However, it is to be hoped that such an activity will also promote thinking about relationships and so assist the learning of selected mathematical concepts. The answers to these problems are left to the reader, but notice how the last three exemplify the need for skills of estimation. If trial and error is used randomly on the first of the 'Missing Operations', it may be necessary to try as many as 4 × 4 or 16 possibilities before the correct answer is found. The chosen numbers are purposely large ones, so as to offer a challenge, but it will be noticed that this type of activity lends itself to similar work at any level. For example, in the early primary years they may involve numbers only up to, say, 30, and only one or two operations. The child can then work with his intuitive or 'counting' understanding of place value, since exercises of this type would not require knowledge of the formal algorithms for the operations. Here is such an example:

$$\square\,8 - \square = 13$$

A more complete discussion of this particular type of activity with a calculator is given in Johnson (1981).

There is another aspect of 'number sense' that is also very critical and is part of basic numeracy. This is the ability to assess the 'reasonableness' of a result (Johnson, 1979). However, a major difficulty in teaching about 'reasonableness' stems from the lack of definition or of precision in the word itself. What do we expect a person to be able to do? Consideration of this point leads to the identification of two different types of 'reasonableness'. We expect people to react to a result, or to a number, either in terms of physical reality, that is, to relate numbers to experience and what one knows about the real world, and/or in terms of selected mathematical relationships. A brief example of each of these will illustrate what is meant.

In the first case, suppose one is asked to calculate the 'miles per gallon' for a new automobile, given that the driver drove 275.5 miles on 8.2 gallons of petrol. Here, without looking at the numbers, everyday experience would suggest an answer between 20 and 40 miles per gallon. So, if the result of the computation gave something like 3.36, an error should immediately be suspected. In the second area of 'reasonableness', the child should exhibit an ability to relate a result to what he or she already knows about numbers and about number properties. For example, if a problem implies having to find the sum of two positive fractions, say, 1/2 and 1/3, the child should note that the sum must be greater than the larger fraction. So, a result of 2/5 (the most common error and one which is less than 1/2) evidently warrants another look. It is not difficult in the regular mathematics programme to provide instruction on the above. But this instruction needs to be better planned. It is typically the case today that children are exposed to such instruction 'after the fact'. The teaching, in other words, comes in a discussion of particular errors a child has made. So the instruction and practice (if any) are laid on top of an already incorrect response to an exercise. It is necessary to question this on pedagogical grounds, as the child is left trying to grasp two very different ideas. One is that of correcting the response to a particular question. The other is learning techniques for assessing reasonableness. In addition, a child who hasn't made the error may not even receive the instruction on the 'reasonableness of results'.

A framework for classifying calculator computer activities

Let us now turn our attention to the more practical aspects of what teachers might incorporate into instruction in primary-school mathematics. In selecting activities, one can use a framework or set of descriptors and classify each activity in terms of its main purpose. It is not intended that a given activity must fit neatly under one of the descriptors. The descriptor, rather, provides a guide for the teacher in planning for the use of the activity. For some groups of children, the same activity may well be classified differently for different individuals, as 'readiness' or 'pre-requisite knowledge' will often dictate which descriptor is best. The important idea is that the teacher should reflect on the role of the activity for each individual child in the class. The three categories I use are: concept reinforcement, concept demonstration, and problem-solving (or algorithm design).

Concept reinforcement has the feature that a concept or an idea that has already been introduced or taught is used or embedded in the activity. The use and the data so generated (the output) reinforce what the child has learned. Concept demonstration implies that the procedure for input and processing are probably known to the child. So they may also involve some aspect of concept reinforcement. It is the output that illustrates an important mathematical relationship. In studying the data that are generated, the child discovers, or observes, an important mathematical relationship or concept.

Problem-solving, or algorithm design, is just that. The calculator or the computer offers a tool for children to use to explore, to test and eventually to formulate a solution to a problem. In this case, a 'problem' is taken to be a new situation, or one for which the child does not have a pre-determined method of solution. Deliberation takes place, and the problem-solver may try a number of different approaches or 'heuristics' in seeking a solution. Problems can be further divided into those of a purely mathematical nature and those that involve applications of mathematics. It is certainly the case that both of these can now be easily included in the primary-school mathematics curriculum. The availability of computational aids, calculators and computers means that the child is freed from formidable calculations. He or she can give time to thinking about what to try next and to reflecting on what the output means. For a very exciting discussion of 'real problem-solving' in school mathematics at all levels, see Burkhardt (1981). This is an excellent source of problems and represents an even more diverse approach to the ideas espoused by George Polya. Enough has been said about background and ideas. Let us look at some examples. Here, it is possible to include only a few representative examples. For further suggestions of classroom activities, the reader is referred to the ERIC Calculator Information Center[1] in the United States. This is an excellent source both for bibliography and for reviews of books and articles. Another source is the Schools Calculator Working Party (SCWP)[2] in the United Kingdom.

Examples for calculator/computer use in primary-school mathematics

Concept reinforcement

Consider the situation in which the child has already been taught about multiplication and is ready to look at sets of multiples. It is clear that the calculator/computer can be used to generate the sets quickly either by multiplication or by repeated addition (for multiples of, say, 7, start with zero and add 7 to the previous value, say, 20 'times'). A look at the output will confirm that it is consistent with expectations. Also, the child can study the data and look for patterns. For example, all multiples of 5 end in 0, or 5; why is this so? Another important idea is that of contrasting the two procedures for obtaining the multiples. Is one of them more elegant or efficient? If you have a calculator, which technique, that of multiplication or addition, would you use if you wanted (a) the first 15 multiples of 9, or (b) the 17th multiple of 9? The idea of selecting a best or most efficient technique or algorithm (procedure) for a particular task is an important part of information processing or 'informatics'. The idea is not new. Many convincing examples and arguments are given in Engel (1979).

If the child uses the computer to generate the output, a precise specification of the entire process is needed. What is meant by 'generate multiples'? Two different Basic programs for multiples of 9 are given below:

1. The Ohio State University, 1200 Chambers Road, Columbus, Ohio 43212, United States.
2. c/o The Shell Centre for Mathematics Education, The University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom.

Multiplication

```
10 FOR N=1 TO 20
20 LET A=N*9
30 PRINT N,A
40 NEXT N
```

(* means 'times')

Addition

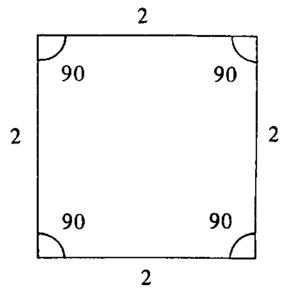
```
10 LET N=0
15 LET A=0
20 LET N=N+1
25 LET A=A+9
30 PRINT N,A
35 IF N=20 THEN STOP
40 GO TO 20
```

The reader is not expected to be able to 'read' these Basic programs, but notice how important it is to be precise. It is also the case that mathematics in a calculator/computer environment is a more dynamic and process oriented subject.

Concept demonstration

Consider the situation in which a computer language has some very 'transparent' commands, such as FORWARD; BACKWARD, RIGHT, LEFT, PEN-UP and PEN-DOWN, and where the first four of these commands are followed by numbers. In the case of FORWARD and BACKWARD, the numbers will specify the units of movement in the direction indicated. With RIGHT and LEFT, the numbers cause turns and they usually specify the number of degrees of angle to be turned. Such commands move an object, either on the floor or on the computer screen. The words are considered to be 'transparent' because they relate naturally to the child's world and to his or her understanding of language. The child might now attempt the problem of finding a way of moving the object, using four forward moves (each move to be the same distance) and three turns, so that the object ends up in the position it started from. After trial and error, the child might develop something like the following:

```
PEN-DOWN
FORWARD 2
RIGHT 90     (degrees)
FORWARD 2
RIGHT 90
FORWARD 2
RIGHT 90
FORWARD 2
PEN-UP
```



It is evident that there are many answers to this problem. Another is: two FORWARD moves, one followed by the other, then a turn of 180 degrees, then two more FORWARD moves. However, in the procedure given above, the child has created or discovered a square. This is a dynamic creation, not a definition. If the child is already familiar with a square, such an activity could be used for concept reinforcement.

In the computer language Logo this type of activity is a natural extension of the child's world. The words used in the above example have in fact been taken from Logo 'Turtle Geometry'. There are many more important features in this language. A particular one is the facility to REPEAT lines and, even more important, to NAME a procedure and CALL (use) it later in another procedure. For example, 'let us NAME our example SQUARE. Now make a TRIANGLE. Can you combine these two to make a HOUSE?' The potential of such a dynamic and exploratory role of computing is exciting. It affords a totally new dimension to the study of mathematics, particularly in the use of computer-generated graphics. For a more complete (and possibly controversial) discussion of the use of the Logo Turtle in primary schools, the reader is referred to two excellent books. One, by Papert (1980) is a general and 'easy to read' discussion of the philosophy with many examples. The other, by Abelson and di Sessa (1981) is a more rigorous mathematical treatment of this computing environment.

One can also use a calculator to demonstrate concepts. The example that follows is taken from Johnson (1978). One key feature of 'concept demonstration' with calculators is that the child can generate many examples quickly. He or she can then concentrate on what is to be learned, rather than looking at a number of seemingly less related examples and wondering where the lesson is going. Consider teaching the multiplication of decimals. A traditional approach is first to explain the rule (for decimal placement) with fractions (a topic children already find difficult). Then exercises are assigned to practise the rules (with most of the time spent doing the usual arithmetic sums). As an alternative, consider the situation in which the child has already been introduced to decimals through fractions with denominators 10, 100, etc. Suppose, too, the child has done some addition and subtraction. The child is now ready to look at multiplication. With a calculator, the following activity for introducing the topic could now be used:

Use your calculator to find these products:

62 × 0.2	0.02 × 0.34
0.8 × 0.6	2.11 × 1.22
3.2 × 0.8	0.72 × 0.6
2.2 × 6.4	0.026 × 0.003

What do you observe about the placement of the decimal point in the answers? Can you make a rule? Why do you suppose this is true?

The class can now discuss their observations and come to some conclusions. At this point, the teacher can discuss, or let the children justify their conclusions using a few easy fraction examples ($\frac{1}{10} \times \frac{1}{10}$,

$\frac{1}{10} \times \frac{1}{100}$, etc.). It should be noted that, in practice, it would be usual to give the children between 15 and 20 items to calculate. It is also necessary to avoid items that would generate a zero in the last digit of the product, as would happen if one factor ended with a 5 and the other had an even number of digits. In the light of this *caveat*, a nice extension of the above activity is to provide some additional exercises of the following form:

Try the following with your calculator:

2.44 × 0.35
1.26 × 0.45
3.60 × 0.40

Does your rule still work? Why or why not?

This activity can be used to reinforce the idea that the rule still holds, as the calculator suppresses the final zero(s) in the result. This reinforces (or demonstrates) the idea that one-tenth is the same as ten-hundredths, etc.

Problem-solving and algorithm design

This is really what mathematics is all about. With today's technology, one must ask 'what is mathematics?', and also 'what is the relationship between mathematics and computer science?' The discipline called 'computer science' is relatively new. In its early days, during the 1950s and 1960s, it was usually called 'numerical analysis'. Donald Knuth, one of the 'giants' in the field of computer science, published a paper in 1974 entitled 'Computer Science and its Relation to Mathematics'. This paper makes the important point that 'computer science', or 'informatics', relies on algorithms as the central core of the discipline. (The word 'algorithm' or 'algorism' comes from the name of a ninth-century mathematician, Al-Khowarizmi. He described step-by-step procedures for solving certain kinds of equations, working with Hindu-Arabic numerals.) Knuth (1974, p. 323) defines an algorithm as 'a precisely-defined sequence of rules telling how to produce specified output information from given input information in a finite number of steps'. Notice that a computer program is a particular representation of an algorithm. Knuth also points out that the study of algorithms is a branch of mathematics, or, given this as a core topic, mathematics is a branch of computer science. In either case, this view suggests that aspects of algorithms or algorithm design should be part of school education. This, in effect, means part of school mathematics.

An important aspect of algorithm design is that, while the end product is 'a precise description of a procedure', the activity undertaken to reach this goal may be very similar to that of the problem-solving process, as Polya describes it. Thus, one may try something, consider the result, make modifications if necessary, and repeat the process until a procedure or an algorithm is produced, which accomplishes the desired task.

Problem-solving, or algorithm design, is not just a final product. It rather includes the process and analysis that is involved in reaching this goal. It is not necessary that an algorithm be a completely machine-based procedure. It may also include a number of extraneous decisions. Nevertheless, these decisions are still part of the algorithm.

What can primary-school teachers do to promote the processes of problem-solving, and algorithm design, in the regular school mathematics curriculum? One answer to this is to provide activities that involve exploration and generalization. The generalization may come from analysing data generated by the algorithm, or it may actually be the algorithm itself.

One potential activity is to provide settings that require children to develop and describe a procedure. For example, consider the situation in which a child has learned that a prime number is a number whose only factors (or divisors) are 1 and the number itself. In a calculator/computer environment, the next logical task might be to ask the child to design a procedure to check whether or not a given (input) number is prime. After analysis and testing, the child's algorithm, or, in this case, the child's list of directions, would probably include some of the following:

A procedure for testing whether a number is prime:

STEP 1. Select an INPUT number (n)

STEP 2. Divide n by the smallest prime (2).
Divide n by the next odd number until this number exceeds x (The value for x will vary, depending on the child's insight. It may be (n-1) or even n) and if this exceeds x, go to STEP 4.

STEP 3. Check the remainder in the division (is there a decimal part). IF no remainder THEN PRINT the number is not a prime number and STOP, ELSE go back to STEP 2.

STEP 4. PRINT the number is a prime number.

Another example of a problem-solving activity with a calculator is given below. This problem will quite likely involve some trial and error followed by a study of the data—a search for a pattern or a relationship, and then a generalization.

You are given the digits 1, 2, 3, 4, 5 and 6. Use all of these, each only once, to make two numbers that give the largest product. For example, one guess is 56 × 1234 = 69104. Can you find a larger product? Can you describe a procedure that will enable one to find the largest product for 1, 2, 3, 4, 5, 6, 7 and 8 (without 'trial and error')?

A nice extension of this activity is to find the smallest positive difference, again using the digits 1 to 6. Also, with each of these problems, the teacher can motivate the children by writing on the chalk board the 'best' answer, 'so far'. The 'best' answer will be replaced by better answers, as they are found by individual children. Eventually the class will discuss why the 'best' answer is the best. This will involve a consideration of place value.

Problem-solving applications are available in many different contexts. However, for an activity to be truly problem-solving, it should involve some decision-making and a justification of its results.

If a child has had some experience of using the computer and the Basic programming language, the opportunities for problem-solving and algorithm design are abundant. For example, consider the problem: 'Is there a pattern in the number of prime numbers in each century (1–100, 101–200, 201–300, etc.)?' Is there a century without any primes? If so, are there any primes in the following century? How many? This is a quite open-ended and fascinating mathematical exploration. The design of the necessary algorithm and its implementation on the computer is very much a problem-solving activity.

Discussion

What does all this amount to? What is informatics in primary-school mathematics? The answer is that using mathematics can be viewed as input, process and output. But, even more important, such work should lead to decisions based on an assessment of the output, and its interpretation. Machines are available to process data. Humans interpret the results. Algorithms processed by machines produce information for people to use.

A second question, and one that is just as important to teachers, is 'what should be the content of school mathematics in a technological society?' Because technology is continually changing, the answer can only be found in the crystal ball.

However, it is fair to say:

1. Mathematics is not the memorization by rote of skills for multi-digit calculation.
2. Mathematics is a dynamic, process-oriented discipline.
3. Problem-solving and algorithm design represent the core of mathematical activity.

Knuth (1974, p. 327) says 'a person does not really understand something until he can teach it to a computer, i.e. express it as an algorithm'. This assertion reinforces my own belief that mathematics is a dynamic, process-oriented discipline. A set of multiples is not just a given static entity. It is rather a process for generating any desired set.

'Primeness' is not just a definition, but rather a means, an algorithm, for testing whether or not a given number is prime. 'Square' or 'squareness' is not just a definition or an attribute. It is rather a process for constructing a square of any size. Estimation is not just a 'best' guess. It rather involves taking a decision about what constitutes a reasonable check on the calculations for a given situation. One could continue. But these few examples should make clear what is inherent in the study of that exciting and very open-ended discipline we call mathematics. Mathematics, then, is both a discipline worthy of study in its own right, and also a tool for investigating relationships in other disciplines. But—and this is to repeat for the sake of emphasis—mathematics is not a collection of techniques. It involves rather the use of processes and techniques to produce data (output) that will assist in decision-making. It is this view that must be instilled in children. And when it is so taught, mathematics will contribute significantly to the objective of 'learning how to learn'. To end, I quote from a poem written by the Danish mathematician Piet Hein (1966, p. 34):

The Road to Wisdom

The road to wisdom—well it's plain
and simple to express—
Err
and err
and err again,
but less
and less
and less.

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Ken Clements

The origins of conceptual difficulties that young learners experience in mathematics

Introduction

The so-called 'conceptual difficulties' in mathematics experienced by infants and primary-school children often arise because parents and schools expect children to learn too much mathematics too quickly. Throughout the world, many primary-school children are expected to learn topics in mathematics that they are not yet ready to master. Hence, difficulties with mathematical concepts often have their origins in inappropriate mathematics curricula.

Sometimes, however, children fail to learn mathematical concepts that they appeared to be ready to learn. While this can, and often does, arise because inappropriate teaching methods have been used, it can also be the result of a whole range of personal factors associated with individual learners. For example, a child may have been absent from lessons when an important concept was presented. Another child, although present at the lessons, may have been daydreaming. This essay will not attempt to take account of personal factors such as these. It will concern itself with reviewing the literature on those conceptual difficulties that occur because the learning experiences provided in mathematics classrooms do not produce the necessary links in children's minds.

The purpose of this essay is to review contemporary literature on the origins of conceptual difficulties in mathematics experienced by young schoolchildren. It will concentrate on important issues, and it will set out some points that are reflected in research findings and experience in many countries. Findings of intensive research into small issues will rarely be discussed. And no attempt will be made to summarize current knowledge about each of the diverse range of topics that currently make up primary mathematics throughout the world. Conceptual difficulties will be considered as broadly belonging to two main categories: first, those arising because of factors external to learners, such as inappropriate curricula, the language of instruction and poor teaching methods; and second, those arising because of more internal factors, that is to say factors associated with the cognitive, the affective and the social development of individual learners. The main emphasis will be on external factors, for it will be argued that in these lie the origins of the internal factors.

Before discussing the external and internal factors that contribute to conceptual difficulties, it will be useful to clarify the notion of a 'mathematical concept' and to give possible meanings to such expressions as 'Johnny does not have an adequate concept of subtraction'.

Mathematical concepts and their acquisition by school children

Recently I observed a lesson on the 'volumes of prisms'. This was given to a seventh-grade class in an Australian school by a trainee teacher. During the lesson, I formed the impression that hardly any child seen in the class knew the meaning of the word 'prism', and that the teacher was not aware of this. At the end of the lesson, I discussed the impression I had formed with the trainee teacher, and reinforced the points I was making by putting questions to two of the children who had been in the class.

After a few preliminary comments, I showed the two children a model of a tetrahedron and asked them if they thought it was a prism. They replied: 'Yes, because it is a three-dimensional solid.' I then showed the children models of a cube and a rectangular prism and asked them if these were prisms. One said they were 'because they're both three-dimensional'. The other said that the rectangular prism was a prism, but the cube was not 'because it's a cube'. I then asked the children what they thought a prism was. Neither felt confident enough to give a definite answer, although both agreed that a prism had to be a three-dimensional solid. When I asked them if any three-dimensional solid is a prism they said 'Yes', although one of the children thought that a cube might be an exception to this rule. A comment made by one child was especially revealing: she indicated that she regularly watched a national television quiz programme, *Pick a Prism*. In this, successful participants are invited to 'pick a prism', from a number of identical 'prisms' (within each prism a description of a prize is given). This child's concept of a prism was linked to the visual examples of prisms presented on television. It is somewhat ironic, and disturbing, that, although the so-called prisms on the television quiz show are complex polyhedra, they are not, in fact, prisms.

The discussion with the two children points to several matters pertinent to how children acquire mathematical concepts. The first is that a concept is an abstract entity. While there can be different concrete embodiments of the same concept, it must be recognized that any one of these concrete embodiments is not, in itself, the concept. The two children appeared to recognize this because they thought that any three-dimensional solid (except, perhaps, a cube) is a prism. The recognition that concepts are abstract, not concrete, entities has important implications for teachers of primary-school mathematics. All too often theories developed by psychologists (such as Jean Piaget) or by mathematics educators (such as Zoltan Dienes) are used to justify the view that since primary-school children are at the 'concrete operational stage' they are not capable of learning mathematical concepts unless they are actively involved with concrete learning aids. One of the problems with such statements is that they do not specify clear criteria for the selection of satisfactory concrete learning aids. Vagueness on this point has sometimes led to the over-use of particular learning aids in school mathematics, Cuisenaire rods, for example, were used in infant and primary mathematics classes in many countries during the period 1950-75. And in some they were used to such an extent that the study of mathematics became known as 'Cuisenaire'. Cuisenaire rods, of themselves, are potentially excellent teaching and learning aids. But any attempt to teach mathematical concepts largely by means of a single concrete embodiment will inevitably result in some children becoming very familiar with the embodiment but not grasping the underlying concepts they are meant to illustrate. On the other hand, numerous attempts to apply Dienes's multiple embodiment principle (Dienes, 1971) in primary-school classrooms around the world have drawn attention to the confusion that can arise if too many embodiments of the same concept are presented to young children. Certainly, since concepts are abstract entities, it is sensible to attempt to assist their acquisition by presenting a number of 'rich' embodiments. But the teacher needs to be alert to the possibility that young minds can learn all about the embodiments but not come to an understanding of the concepts.

It should also be emphasized that since only a tiny proportion of primary-school children can formalize, to themselves, the abstract nature of even the most elementary mathematical concepts they are being asked to learn, important mathematical concepts will almost inevitably be associated in young children's minds with particular embodiments of those concepts. Thus, it is important that embodiments presented in schools should be rich. That is to say, they should contain features that are easily linked with the relevant concepts, but should exclude features irrelevant to the concept that are likely to attract too much the attention of young learners. This leads to a second matter arising from the discussion of 'prisms' with the two children. It is important for a teacher to identify all the characteristics that define a concept, and to make available concrete examples in which all the defining characteristics are obvious. Thus, in the case of a prism, the requirement that it should have congruent ends, corresponding points of which are joined by parallel line segments, is as central to the geometrical concept as is the three-dimensional requirement. And, since the three-dimensional characteristic of a solid is perceptually obvious, a teacher wishing to communicate the concept of the prism should emphasize the other defining characteristics.

Once the defining characteristics of a mathematical concept have been identified, infant or primary-school teachers need to emphasize to their students that anything that possesses all of the characteristics is an exemplar of the concept. Thus, for example, a cube is not only a cube. It is also a prism. It is usually a good teaching strategy to draw attention to non-exemplars of concepts. Thus, for instance, neither a tetrahedron nor any pyramid is a prism.

It would be foolish to expect young children to learn precise verbal definitions of mathematical concepts. And, even when, at a later stage, it is reasonable to expect children to know the definitions, it should be recognized that, in most cases, a child's cognitive structures of the concepts will contain much more than the relevant verbal definitions. In particular, the response of the child to a task involving a particular concept is likely to be guided by recollections of real-life episodes associated with the concept in the child's mind and by images of the concept that can be evoked (Gagné and White, 1978) and by the links these provide with the verbal definition. Vinner and Herskowitz (1980) provide a nice illustration of this point when discussing children's concepts of an isosceles triangle. Their data show that many children have a concept image of an isosceles triangle similar to that shown in figure 1(a), where the base is horizontal, but their concept images do not allow for the possibility that the triangle in figure 1(b) is an isosceles triangle. While such children might be able to provide an accurate verbal definition of an isosceles triangle, they would not agree that the triangle in figure 1(b) is isosceles.

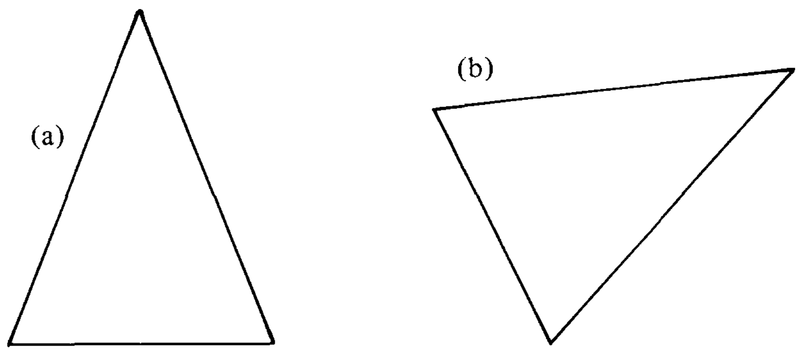

Fig. 1

In the light of the above discussion of the nature of a concept, it is interesting to consider the meaning of a common expression such as 'Johnny does not understand the concept of subtraction'. Is there such a thing as the concept of subtraction? Presuming that for infant and primary-school children the arithmetical operation of subtraction is confined to pairs of natural numbers and zero (and to examples where the answer is a positive number or zero), it seems intuitively clear that children who have learnt, by heart, all their elementary subtraction facts (e.g. 7−5=2), have mastered an appropriate subtraction algorithm (e.g. for finding 752−384), may still not possess a satisfactory concept of subtraction. A satisfactory concept of subtraction for natural numbers would appear to involve not only the notion of the difference between two numbers, but also the ability to recognize familiar situations in which the subtraction operation should be carried out. Just as it would be inconceivable to say that someone who could not be persuaded that a cube was a prism had a firm grasp of the concept of a prism, it would be just as wrong to suppose that someone who did not recognize simple situations in which the subtraction process would be useful could have an understanding of the concept of subtraction. A similar comment would apply to the concept of a rational number, and to many other mathematical concepts commonly associated with school mathematics.

If, indeed, the notion of a mathematical concept is extended, so that it necessarily involves the ability to recognize exemplars of the concept, then the importance of identifying conceptual difficulties and taking steps to assist teachers to create learning environments that will minimize these difficulties becomes an urgent matter. With arithmetical operations, for instance, the main concern must not be whether children know their elementary number facts and appropriate algorithms, or whether they know which keys to press on an electronic calculator in order to find the answer to an arithmetic calculation. Rather, it is important to ascertain if the children know which particular operation or sequence of operations should be used to resolve a given situation. And, as recent error analysis research has demonstrated (Clements, 1980), many children who know their elementary number facts perfectly, and can perform arithmetical operations by standard algorithms accurately and reliably, have difficulty deciding which arithmetical operation should be used to obtain answers to apparently simple arithmetical problems presented in words. If the sense in which the term 'mathematical concept' is being used in the present paper is accepted, then such children must be regarded as experiencing difficulties with basic mathematical concepts.

External factors contributing to conceptual difficulties in mathematics in primary-school children

Curriculum

In a recent study, by Glen Lean and myself, into factors affecting mathematical learning in Papua New Guinea it was concluded:

> The needs of mathematics education in Papua New Guinea would, perhaps, be better served by syllabuses which took more account of the immediate backgrounds of pupils and teachers, and were pitched at levels at which the pupils really are, rather than at those levels which curriculum planners believe they ought to have reached. (Clements and Lean, 1981.)

While these are strong words, they certainly accord with the writings of curriculum theorists, such as Skilbeck (1976), who emphasize the need for curriculum goals to be based on situational analyses. It is futile for mathematics curriculum planners to continue to prescribe for all schoolchildren at certain levels extensive work involving, for instance, operations with fractions, or long division, or axiomatic proof with a Euclidean geometry or a framework of abstract algebra, if it is clear, from research, that the majority of schoolchildren are not ready to learn such topics. While it is possible to justify the inclusion of such topics in a curriculum because of the need to prepare children for higher mathematical studies and a desire to make children aware of the axiomatic nature of much of mathematical thought, there is the real danger that such prescriptions will suit only the gifted few and that the others will learn, from their failure to understand teachers and from their failure to pass examinations, that they cannot do mathematics.

Not only do mathematics curricula for schools in many countries (not only developing countries) prescribe topics that are too difficult for many children at the given levels, but they often fail to take account of the peculiarities of the environments and the cultures in which the schoolchildren live. This raises the question of the extent to which mathematics curricula can be, and should be, independent of culture. The Chinese experience, during the late 1960s and the 1970s when the education authorities encouraged the development of school mathematics curricula based on the needs of localities and of the state (Swetz, 1974), suggests that persons attempting to localize mathematics curricula must be prepared to face large and unpredictable difficulties.

The question of whether the teaching of mathematics corresponds to the needs of pupils and the society was specifically the focus of Volume 2 of *Studies in Mathematics Education* (Unesco, 1981). It will not therefore be considered in detail here. A few comments may be helpful, however. The first comment is that if primary-school mathematics curricula of developing countries continue to be based on

Western models, then, although it is likely that children taking the courses will ultimately perform at the same levels as Western children on comparable pencil-and-paper tests, this achivement may take time and will cost the countries dearly in terms of loss of traditional culture. Pam Harris, in her recent and provocative study *Measurement in Tribal Aboriginal Communities* (Australia) has made some pertinent comments on the conceptual difficulties young learners experience as a result of such a policy. She has argued, with the support of relevant data, that

> any applied number program which is written for English-speaking white Australian children living in urban situations is likely to be both inappropriate and inadequate for use with vernacular speaking Aborginal children living in remote tribal communities, because:
>
> many of the concepts presented will be foreign to the Aboriginal child and in conflict with his traditional world view;
>
> in many instances the way of expressing a concept in the child's first language will be quite different from the way in which it is expressed in English, thus causing confusion with vocabulary and terminology;
>
> in some cases, where a concept is totally foreign to the child's culture, there will be no concise way of explaining it in the child's own language; thus, the child will be required to learn new vocabulary and a new concept, both at the same time; and usually to do so in a second language;
>
> many of the concepts introduced will not be reinforced outside of the school because they are either not used in the Aboriginal community, or are in conflict with established custom;
>
> the concepts presented will assume prior knowledge and experiences that the Aboriginal child does not have, while ignoring the different set of knowledge and experiences that the Aboriginal child does have;
>
> failing to take account of the Aboriginal child's different cultural environment, such a program will also fail to observe the fundamental principle that teaching must always begin with that which is known and proceed from the known to the unknown. (Harris, 1980.)

Clearly, these points were meant to apply to a particular situation. However, with suitable modifications, they could more or less apply to any situation where a curriculum defined for one culture is used as a basis for developing a curriculum for another culture. It could even be argued that the points made are still relevant in certain cases where the two cultures share the same first language. This is because language usage within and across cultures can differ in important ways, even though, superficially, it appears that the same language is spoken (for example, the language patterns of an élite urban group may differ from those of a less educated, rural group).

It is a corollary of Pam Harris' argument that the modern idea of a 'core' mathematics curriculum, containing only that which is essential for survival with dignity in Western society, represents little more than a value-laden excuse to impose, however subtly, Western ideas on everyone. Pam Harris provides data to show that the alleged need to learn Western concepts is not always accepted by non-Western children. Thus, for example, only a tiny proportion of 14-year-old tribal Aborginal children in Australia could tell the time from a dial clockface, despite the fact that it was assumed in the mathematics curriculum they had followed at school that they had learnt to tell the time by the age of 7 or 8 years. Why should persons whose traditional life-styles are not dependent on knowing how to measure time accurately (with Western instruments) be required to learn to tell the time? Similar questions could be asked, of course, with respect to the acquisition of many other so-called basic mathematical concepts.

Language factors

Another factor external to students, but affecting their grasp of mathematical concepts, is the language of instruction used in the mathematics classroom. It is common in the literature of mathematics education to distinguish between the learners who are taught in their own language and those who are not (Austin and Howson, 1979). This distinction will also be made here, with the focus of the following discussion being mainly on the plight of children who do not receive instruction in mathematics in their first language.

Bryan Wilson (1981) has provided recently an excellent summary of the policies for language in education that have been adopted in different nations. In some countries, young children, on their commencing school, receive instruction from the outset in a language that is not spoken at home. In other countries an attempt is made to teach the children in the mother tongue for the first few years of schooling, but then national languages, which the children may not use in out-of-school hours, are used for teaching higher grades. In yet other countries, the vernacular languages of children are used as much as possible throughout their schooling, even if this means that the official 'national' language is not used in school. According to Wilson, decisions on the language of education in school are usually made for political, not educational, reasons.

Wilson (1981, p.160) cites an interesting recent longitudinal study, conducted in primary schools around the University of Ife, western Nigeria. This was into the question whether pupils learn better if they are taught in their mother tongue through the whole of their primary education, or if they learn better by changing into English at the level of Primary 4. The latter was national policy, but the university was granted permission to conduct the experiment in some schools, using Yoruba (the name of both the people and the language in the region) for the language of instruction for the experimental classes, throughout primary school. The materials were written in Yoruba, translated into

English for half the schools, and left in Yoruba for the other half. The same examination was set at the end of Primary 7. This was written in Yoruba and translated into English for those who had been using English. Results indicated that all students who had been taught in their mother tongue did uniformly better on all tests given. While the experimental design of the study is open to criticism (e.g. the Hawthorne effect was not controlled), the result, in terms of the cognitive development of children, appears to have important implications for language policies in schools throughout the world.

Would a country's policy on the language of instruction affect young children's ability to grasp mathematical concepts? Undoubtedly yes. One only has to examine courses prescribed for infant and primary-school mathematics throughout the world to be convinced that the children's comprehension of language, and especially the language used to teach mathematics, is of crucial importance (Hollis, 1981). An investigation carried out by Peter Jones (1981) in Papua New Guinea, provides an excellent illustration of the type of conceptual difficulties that can arise as a result of an inadequate comprehension of common mathematical terms.

Jones studied the children's understanding of such expressions as 'more', 'more than', 'less' and 'less than'. He devised a pencil-and-paper test containing open-ended items in what he termed 'comparative', 'direct' and 'indirect' forms. Some examples follow:

1. Which is more, 10 or 13? (Comparative)
2. Which is less, 7 or 9? (Comparative)
3. What is 3 more than 6? (Direct)
4. What is 3 less than 5? (Direct)
5. The number 8 is 2 more than which number? (Indirect)
6. The number 6 is 2 less than which number? (Indirect)

Jones gave his test to children in grades 2 to 10 in community schools and in provincial high schools in Papua New Guinea, and to children in grades 2 to 6 in international primary schools in Papua New Guinea. His data showed clearly that, whereas 'comparative' items were mastered by the third year of schooling by both community-school and international-school children, the international-school children mastered the other two forms much more quickly than community-school children. 'Direct' items had been mastered by almost all international-school children by the fifth year of schooling, but it was not until the seventh year of schooling that this was the case for Papua New Guinea children attending provincial high schools. 'Indirect' items had been mastered by almost all international-school children by the seventh year of schooling, but this could not be said of even grade 10 students in the provincial high schools.

Error analysis carried out by Jones revealed that the most common errors which Papua New Guinea children made on 'direct' items were the result of their having used a 'comparative' strategy. Thus, common answers to the question 'What is 3 more than 6?' were 'No', and '6'. Similarly, the most common errors made by Papua New Guinean children on 'indirect' items were the result of their having used either a 'comparative' or a 'direct' strategy.

It is not surprising, of course, that Papua New Guinean children's knowledge of English terms and expressions is not as good as that of the international-school children, for whom, in the great majority of cases, English is the first language. However, since English is the official language of instruction in Papua New Guinean schools, it is obvious that language difficulties will often lead to conceptual weaknesses in mathematics. In school almost all mathematical concepts are inextricably linked to linguistic concepts. Children who are instructed in a language other than their mother tongue are, therefore, at a disadvantage in learning mathematics. And, all too often, the mathematics courses of study followed in countries where this commonly occurs are based on, and are as difficult as, courses used in developed nations where the language problem is not so acute. In countries where many languages are spoken, there often seems to be no viable alternative to using one major language as the language of instruction in schools, But, in these cases, it is important to ensure that not only is every effort made to make textbook writers, teachers and evaluation bodies more fully aware of the conceptual weaknesses in mathematics that arise from language difficulties, but also that courses of study are not necessarily based on overseas models.

Although Bryan Wilson (1981, p. 164) rightly points out that 'linguistic specialists assure us that any language has the capacity for development to the point where it can bear the weight of whatever demands are made upon it', it does not follow that it is dangerous and misleading to claim that a certain language has deficiencies so far as the teaching and learning of mathematics and science are concerned. I would maintain that all languages are not equally useful for the teaching and learning of mathematics. Such a statement does not imply, however, that I believe that some languages are 'lesser' languages than others. In fact, as Bryan Wilson and Pam Harris (1980, p. 75) suggest, many local vernaculars are often able to express clearly and simply concepts that would require much lengthier descriptions in English or French.

An interesting illustration of the point being made occurred in the investigation which Glen Lean and I carried out in Papua New Guinea. None of the mother tongues of the various groups we investigated contained a word for the rational number 'one-third'. So there was no generalized expression for the operation of 'taking one-third of' something. Thus, the following question (which would have meaning to most children from grade 4 onwards for whom English is the first language)

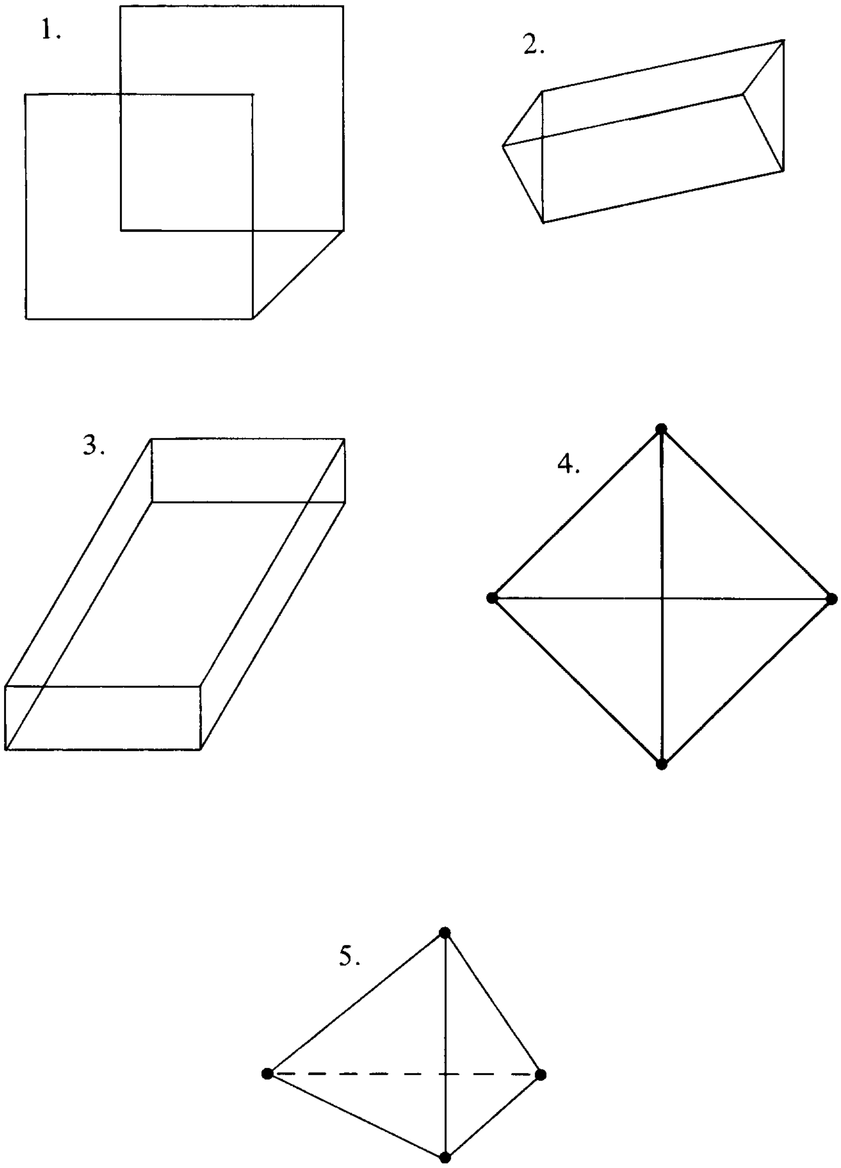

Fig. 2 Five drawing to stick-model tasks (after Deregowski).

could not be translated exactly into the mother tongues of the children: 'Here are six buai. If you gave me one-third of them how many would you give me?' Such a question had to be rephrased to make it meaningful in the mother tongues of the children. Usually, something similar to the following supposedly equivalent question was framed: 'Here are six buai. Suppose you shared them among Jack, Luke and Kuni so that each got the same number. How many would each get?' While the two questions have the same answer, the former is more difficult. This is because it involves the more general form of the concept 'one-third of', whereas the latter avoids this. A consequence of the recognition by mathematics educators of the importance of language in mathematical learning has been the tendency to avoid, in textbooks, in examinations and in classroom teaching, the use of important mathematical expressions. This is unfortunate. While rewording, as in the example shown above, may enable children to obtain a correct answer to a particular exercise, it may also hinder the development of essential concepts because of their evasion.

A very different aspect of language, which contributes to conceptual difficulties in mathematics with primary-school children, is the lack of knowledge of diagramatic conventions. The five-line drawings shown in figure 2 have been used by a number of investigators in different parts of the world (Deregowski, 1974; Bishop, 1979; Clements and Lean, 1981). Children are shown the line drawings and asked to make each of the figures out of sticks and plasticine. In developing countries it is common for primary-school children to produce two-dimensional models. And when responding to the fifth drawing in figure 2 children often ask if they are permitted to break the sticks they have been given into small pieces. Now, there is nothing inherently wrong about such responses. The children who give them have not become aware of certain drawing conventions, which most children in more developed countries acquire during their first few years of primary schooling. Indeed, there is no good reason why the drawings in figure 2 should suggest to children that the required models could be three-dimensional. Mathematics textbook writers and teachers in developing countries should be particularly conscious of this point when they attempt to communicate to schoolchildren geometrical concepts, especially concepts associated with three-dimensional solids.

Evidence continues to accumulate to support the view that differences in mathematics performance between nations and between groups can be explained more easily by social effects than by basic cognitive differences (Kagan et al, 1979, Clements and Lean, 1981; Lancy, 1978). So, children who have not grown up in environments in which line drawings, photographs and audio-visual experiences are commonplace are at a disadvantage compared to others, if they are asked to perform tasks that involve the recognition and interpretation of features in line drawings, photographs and audio-visual presentations (Michelmore, 1980). Similarly, if the language of instruction is English, then children whose mother tongue is not English are at a disadvantage compared to those for whom English is the first language.

Teaching methods

Traditional pedagogical wisdom asserts that in most cases when children do not learn well at school, the failure is as much the teacher's as it is the child's. Paul Trafton (1980), in the course of arguing that this wisdom applies in mathematics education, listed the following six criteria for effective mathematics teaching:

The instruction needs to be developmental, so that new learning is related to familiar knowledge.
The instruction needs to be well sequenced.
The instruction needs to be focused on what students must be able to do as a result of learning.
The instruction needs to promote mental activity by students.
The instruction needs to be cumulative.
The instruction needs to be comprehensive.

Lists of this kind have been presented in articles and in textbooks on teaching methods throughout this century, and nearly every innovation in the mathematics syllabus has been accompanied by recommendations for change in teaching style. Yet who would doubt that James Fey's recent depressing conclusions on mathematics teaching in the United States would not also apply to most countries? In summarizing data on mathematical teaching obtained during a large national survey, Fey (1979) wrote:

> The profile of mathematics classes emerging from the survey data is a pattern in which extensive teacher-directed explanation and questioning is followed by student seatwork on paper-and-pencil assignments. NSF (National Science Foundation) case studies paint a similar, but even more pedestrian, picture of day-to-day activity in mathematics classes at all grade levels. Certainly, the mathematics which the observers saw embodied none of the spirit of inquiry, laboratory exploration, or individualization urged by experts.

He went on to say that the case-studies and the survey produced frequent comments that students find the study of mathematics boring, and that teachers find the students' lack of motivation to learn mathematics one of the most difficult problems they face. When asked with what aspects of their jobs that most needed help, teachers at all levels mentioned learning new teaching methods and implementing discovery/inquiry approaches. However, an extensive review of recent research and expository literature made by the National Science Foundation offered little immediate hope that instructional research will yield new ideas or convincing support for any existing strategy or technique.

What has become clear from research during the 1970s is that the high hopes held for the growth of individualized methods, involving the extensive use of stimulus-reponse programmed learning materials and mastery learning procedures, have not materialized. The brilliant ethnographic study by Stanley Erlwanger (1975) into children's conceptions of mathematics showed how some primary-school children had difficulty with certain mathematical concepts because of inherent weaknesses in the individualized instruction programme in which they were participating. Erlwanger concluded that the programme intended to force children to assume a passive learning role, to inhibit the development of their intuitive ideas, to create conflicting roles for the teacher and the child, and to eliminate traditional classroom practices, such as teacher demonstration and group participation. More particularly, Erlwanger produced evidence to show that the programme, which appeared to meet all the criteria which are usually specified for a good individualized learning system, resulted in children developing a view of mathematics as nothing more than a set of rules for making marks on paper and for producing answers by rigid methods. Any relationships between school mathematics and intuitive ideas about the topics covered were never developed. Procedures were invariably justified purely in terms of rules that had been specified for symbol manipulation. The children tended to resist remedial work in which methods were taught which were different from those originally presented in the individualized materials. Thus, since little attention was given to developing generalized concepts, serious conceptual weaknesses, which were not easily remedied, resulted.

Erlwanger's research served to re-emphasize the importance of constant teacher-pupil interaction if mathematical concepts are to be learnt. As Heinrich Bauersfeld (1980) has written, 'teaching and learning mathematics is realized through human interaction'. According to Bauersfeld, a student's initial meeting with a mathematical concept is mediated through parents, playmates and teachers. The student's reconstruction of its mathematical meaning (which is essential if concepts are to be well formed) is a construction that grows via social interaction about what is meant by the concept and about which interpretation of meaning gets the teacher's (or the peer's) sanction. From this point of view, a major weakness of the individualized systems, which became popular in the 1960s and early 1970s, was their inadequate attention to the teaching and learning of concepts, due to neglect of the interactive constitution of individualized meanings.

Despite the pessimism implicit in the above comments on the possibility of improving concept learning by effecting large-scale changes in methods of teaching mathematics, it is encouraging to note that attempts are being made around the world to produce curriculum materials, accompanied by practical teaching procedures, which will enable children to learn important concepts through experiences consonant with their cultures. A notable case in point is provided by the work of the Indigenous Mathematics Project in Papua New Guinea. The final report of this project, which included substantial inputs from local and overseas mathematics educators, teachers, anthropologists and psychologists, concluded that the prompt introduction of an appropriate, locally-developed textbook for grades 4, 5 and 6, and comprehensive teacher handbooks for grades 1, 2 and 3, would, in conjunction with minimal teacher aids and adequate in-service training, have significant effects on teacher competence and student achievement. Randall Souviney (1980), who headed the Indigeneous Mathematics Project between 1979 and 1981, has reported that results of the trials of the curriculum materials and the in-service have been encouraging. While projects such as this, which aim at substantially improving mathematics education within whole nations, are ambitious, their potential for preventing many children from having unnecessary conceptual difficulties in mathematics cannot be denied. For, as David Lancy (1978) has pointed out, research evidence now seems to confirm that attending school can accelerate the transition to symbolic thought. However, if schoolchildren are denied the opportunity to engage in practical activities because, perhaps, the teacher rigidly controls the classroom and spends a great deal of time lecturing, their cognitive growth will accelerate less rapidly, if at all.

One of the biggest problems that affect the quality of primary mathematics teaching around the world is that too many teachers possess minimal qualifications in mathematics. As well as this, they have negative, and narrow, attitudes towards the subject. In the United States, for example, many elementary-school teachers act as though they believe that their sole responsibility in mathematics teaching is to develop facility in arithmetic computation (Fey, 1979, p. 19). The first step towards solving this problem is for national education authorities to demand higher levels of mathematics as pre-requisites for entry into primary-school teaching training courses. The second step is to compel all trainee teachers to take courses in both mathematics and mathematics method as part of their training courses. The third step is to require practising teachers to attend, from time to time, in-service courses, in which both mathematics and topics in mathematics education are covered in some depth. However, the politics inevitably associated with moves to embody in educational legislation such requirements have often proved too much for mathematics educators in different parts of the world.

Internal factors contributing to conceptual difficulties in mathematics in primary-school children

During the 1970s, the tendency for cross-cultural psychologists to argue that people of different races had fundamentally different cognitive abilities declined, with many researchers (Stevenson et al.,

1978) reporting data that seemed to point towards the conclusion that there are no cultural differences in the basic components of the cognitive processes. So it is unlikely that any cultural group wholly lacks a basic process such as abstraction or inferential reasoning or categorization. Ginsburg, Russel and Posner (1981), in a study of the development of arithmetic skills in African and American children, reported that analysis of protocols of children solving simple addition problems indicated that children from the Ivory Coast experienced the same pattern of difficulties as their American peers. David Lancy (1978, p. 135) and Glen Lean and I (Clements and Lean, 1981, p. 23), in independent studies, which involved the administration of different batteries of cognitive tests to Papua New Guinean children and white children attending international schools in Papua New Guinea, found that when significant differences between groups did occur, these could be readily explained in terms of factors such as language and home background. Hopefully, these and other similar findings reported in many parts of the world will become known to those mathematics teachers, and others, who are convinced that members of certain groups are incapable of carrying out the sophisticated reasoning processes needed for higher mathematics.

Nevertheless, it would be foolish to claim that cultural factors do not play an immensely important role so far as performance in mathematics is concerned. Gay and Cole (1967), for example, suggested that the primary characteristic that prevented Kpelle children from succeeding in school was their unquestioning reverence for knowledge possessed by adults. While such behaviour may be necessary for survival in the environment of the traditional village, such behaviour in school is clearly inefficient. As Randall Souveney (1980, p. 1) has argued:

> Without access to the scientific tool of initially reserving judgement while engaging in a process of objective analysis, the masses of new information encountered in school are destined to remain a mystery of unconnected facts. The task of the teacher, then, is to help the child translate Western mathematical ideas using analogs in their own experience and extend these ideas wherever necessary to define concepts that are not readily expressable in local terms.

Pam Harris would almost certainly wish to qualify this statement. There should be a judicious selection of those Western mathematical ideas that need to be translated into local terms, and the criteria for their selection should not be solely, or largely, the responsibility of Westerners.

Thus, external factors, that is to say, those factors over which an individual learner has no control, often give rise to internal factors that affect the individual's ability to grasp mathematics. If, for example, children grow up in villages where the Western idea of a fixed unit of measurement is largely unknown, then it will take some time for children in these villages to grasp this notion if and when they meet it in primary mathematics classes in their village schools (Jones, 1974). It is not difficult to explain this in terms of modern cognitive psychology. Robert Gagné and Richard White (1978) have proposed that, in attempting to accommodate new information, learners call upon various kinds of information already in their memory structures. Such information is stored in the form of factual knowledge, intellectual skills, imagery and episodes. Both the recall and the intellectual construction and reconstruction needed to grasp a new concept will be facilitated if the stimuli that motivates thinking about the concept help to establish appropriate links between existing information in the children's cognitive structures. It is not surprising, then, that, if children are presented with concepts which, from a relative point of view, are largely foreign to their day-by-day experiences out of school, they will be unlikely to make the necessary links. From this point of view, it is necessary to question the findings of cross-cultural, developmental psychologists, who have often used culturally-biased tasks, such as the Piagetian conservation tasks, which suggest that children in developing countries are often years behind their peers in more developed countries in their cognitive development.

Such findings probably indicate nothing more than that the cognitive structures of these children are not providing adequate links for the tasks given by the psychologists when they are attempting to assess the children's levels of cognitive development. If the concepts implicit in these tasks are important for school mathematics, then one of the roles of the mathematics teachers must be to create classroom environments in which the concepts will be easily learned. Perhaps mathematics educators have taken too much notice of research, of dubious value, carried out by multitudes of cross-cultural developmental psychologists around the world.

During the last two decades, a number of so-called cognitive style variables have been used to explain individual differences in learning style. Witkin's field dependence/independence construct is probably the best known of these, though impulsive/reflective and visual/verbal constructs are other dimensions of cognition that have been used by mathematics educators attempting to explain how children acquire mathematical concepts. It is not possible here to review the extensive literature on these variables in cognitive style. However, although I have used cognitive style variables in my own research, I would wish to warn mathematics educators against their overuse. If we are not careful, we will find psychologists and educators rushing into every corner of the globe administering cognitive style tests, and the data obtained will be just as difficult to interpret sensibly as the data obtained over the past two decades by the host of developmental psychologists who have administered conservation and other Piagetian tasks around the world.

Summary

In reviewing the literature on the origins of conceptual difficulties young learners experience in mathematics, special attention was paid first to the definition of a mathematical concept and the implications of this for the teaching and learning of primary-school mathematics. Stress was placed on the need to clarify the characteristics that define important concepts, and on teachers' making available to children concrete referents, in which all the defining characteristics are obvious. While a number of different embodiments of a concept should be presented to young children, the provision of too many embodiments can lead to confusion. It is usually a good teaching strategy to draw attention to non-examples of concepts, especially to concrete referents that possess some of but not all the defining characteristics.

It was argued that unless children can recognize simple situations in which concepts are embodied they do not understand the concepts. Thus, primary-school teachers should be at least as concerned that their students know when to apply concepts, in familiar situations and uncomplicated unfamiliar ones, as they are to ensure that they know elementary number facts and algorithms.

The discussion on the origins of conceptual difficulties was divided into two sections. The first and major section concerned external factors—factors that affect students' understanding of concepts but are completely outside their control. The second section was concerned more with internal factors—factors associated with the cognitive, affective and social development of individual learners. The main emphases in the discussion of external factors were on curricula, the language of instruction, and teaching methods. The view was expressed that in most countries, mathematics curricula for primary schools prescribe topics that are too difficult for many children, given their level of maturity. More particularly, it was argued that curricula in developing countries have too often been based on Western models, and that this policy not only leads to conceptual difficulties but also threatens to cost the countries dearly in terms of loss of traditional culture. Indeed, the modern idea of a core curriculum in mathematics represents little more than a value-laden exercise to impose, however subtly, Western ideas on everyone.

A nation's policy on language of instruction affects young children's ability to profit from mathematical instruction. However, there is no universal answer to this language problem. This is especially so since policy decisions on the matter are usually made for political, not educational, reasons. An unfortunate result of the recognition by mathematics educators of the importance of language considerations in mathematical learning has been a tendency to avoid, in textbooks, in examinations and in classroom teaching, the use of important mathematical expressions. This has resulted in children not knowing vital mathematical terminology; it has also hindered their acquisition of mathematical concepts.

Classroom experiments with different teaching methods have produced disappointing results. There is no evidence that children of today absorb mathematical concepts more readily than did children in the past. In particular, the early promises of programmed learning, and individualized mastery-learning schemes, have not materialized. It seems that human interaction is the indispensible key to the successful teaching and learning of mathematical concepts.

Despite constant reminders from methodologists that young children need to be actively involved in learning mathematics, it is common in upper primary grades throughout the world for mathematics classes to be teacher-dominated. Concrete embodiments of concepts are rarely available. Often the emphasis is almost exclusively on drill, with most attention being given to number facts and to arithmetic algorithms. A healthier state of affairs often prevails in mathematics classes for the youngest children. It is encouraging that some nations have taken steps to provide primary schools with materials that could improve the teaching of mathematics in upper primary grades (for example, Papua New Guinea, through its Indigenous Mathematics Project).

Three recommendations aimed at improving the teaching of mathematics in primary schools were made: higher-level passes in mathematics should be required of intending primary teachers; all trainee primary-school teachers should be compelled to take courses in both mathematics and mathematics method as part of their training courses; and practising primary-school teachers should be required to attend, from time to time, in-service courses at which mathematics and topics in mathematics education are covered in some depth.

Finally, in considering the internal factors that contribute to conceptual difficulties, it was emphasized that internal factors often had their origins in external factors. The most recent cross-cultural research suggests that there are no cultural differences in the possession of basic components of the cognitive processes. It would seem unlikely that any cultural group wholly lacks a basic process such as abstraction or inferential reasoning or categorization. Nevertheless, cultural factors do affect children's readiness to learn mathematical concepts. So it is the task of the teacher to help the children to understand concepts by linking them with experiences with which the children are familiar. In terms of the jargon of modern cognitive psychology, the relevant episodes, images, knowledge and intellectual skills which a child has available in memory need to be drawn out by the provision of appropriate learning experiences. It is likely, however, that only well-trained teachers, who both understand mathematics and are aware of how children think about mathematics, will be capable of consistently providing such experiences.

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Michele Pellerey

Aspects of visualization in the teaching of geometry, and reflections on a case: symmetry in primary school

The process of geometrization

In recent years the attention of didacticians of the primary school has shifted from mathematical knowledge, and theories about teaching it, to the consideration of those cognitive processes that enable students to learn. Correspondingly, there is more emphasis on educational action than on results. Probably it is more correct to keep a reasonable distance between the two perspectives, and to focus one's attention on single mathematical acts.

There seems to be a situational difference between the construction of a geometrical concept and the building up of a concept of a probabilistic nature. The resulting developments are substantially diverse. A. Z. Krygowska, distinguishing between mathematization in the full sense of the word and mathematization at an introductory level, has defined the latter as 'the construction of a mental scheme, concerning certain real relations, which is not yet able to be retained as a true and proper mathematical scheme and therefore taken into a certain mathematical theory, but whose construction from its very beginning is directed towards a true and proper mathematization'. It would be interesting to analyse, in a detailed manner, what factors are at play in this initial form of mathematization: the mental images and, in general, every process of visualization. Such a discourse, however, would become long and complex. So we will limit ourselves to considering the case of the construction of schemes that can be framed, more or less immediately, within the ambit of concepts and theories of a geometrical type. We shall refer especially to concepts of symmetry to exemplify the discussion.

Psychological research has for the last two decades explored the cognitive processes with a certain interest, pointing out, in an often significant manner, the special role of images, either in the long-term (or storage) memory or in the short-term (or working) memory. In particular, in perceptual activity, a schematic form of reality seems to be fulfilled. It is realized under the guidance of patterns already present in the memory, or of patterns brought out by the teacher by focusing attention on some relevant elements or relationships. These patterns selectively direct the attention. They act as a filter of the sensorial stimuli and as a basis for organizing them into a significant scheme. From the mass of information registered by the sensorial receptors there emerge those patterns that are relevant to the problem or to the situation studied. These are then structured in a congruent manner and in an intellectually productive way.

As well as the store of mental images, there exists what has been called the semantic memory. The function of the semantic memory is not only to give a name or a symbol to a perceptual scheme but also to endow it with meaning, connecting it to previous experiences, to possible utilizations, or to other schemes and concepts, either more general ones or more specific ones.

The cognitive process, which from one part of the brain accepts and structures sensorial information and from another retrieves from the long-term memory significant and structuring patterns and then uses them, occurs in the short memory. This means that we deal with an activity in a consciously and intellectually controlled manner. This is equally true when we deal with forms of representation that are coherently organized. If the intellectual schemes and their representations can be led back to geometrical theory (or to a chunk of it), the activity becomes a true and proper activity of geometrization.

Nevertheless, it is important to remember the role and the influence of the meanings that are linked, more or less directly, to these intellectual schemes. Through both family and social experience, and through communication, the child has already accumulated images, interpretations, judgements and uses. All these can be linked with the on-going process of learning. In the activity of geometrization, and above all at the introductory level of this activity, one must take account of that part of the semantic memory of each student that is already organized.

The deep images of symmetry

It is by now common enough to introduce into the primary school either axial or central symmetry. Sometimes, in the scholastic texts, one begins by defining what is meant by saying that two points are symmetrically placed about a line or about a third point. Convenient drawings are used. And the idea is slowly extended to include symmetrical figures and their properties, to the distinction between figures that are directly or inversely equal, to the geometric transformations of a plane and to the composition of more than one transformation.

But how does one link this work with the mind of a child, particularly with his interior world, which he has developed and organized under the influence of family and social communication? Symmetry, it must be remembered, is a deep image, a conceptual category that has dominated and still dominates many cultures. In particular, those cultures–for example, the Italian culture–that derive from Graeco-Roman culture conserve the impression of thousands of abstract elaborations and practical concretizations that cannot be weeded out.

Amongst the first, and the most profound, intuitions of symmetry is the idea of stability as opposed to movement or growth. Stability expresses order, tranquility, rationality, equilibrium. In contrast, movement suggests disorder, anxiety, emotionalism, disequilibrium. The most fruitful image of symmetry is that of a chemical balance, which shows a dynamic equilibrium and symmetry between its parts. To position things correctly with respect to a point of reference is a sign of calm and peace. The wise man will place himself in a balanced position between the extremes. His peace of mind derives from keeping under control his tensions and emotions.

Close to this great category is the idea of moral equilibrium–of personal and social justice. The very same balance is the most diffused symbol of justice. But embodied in this scheme there are also elements of asymmetry. The right indicates reason, good, virtue. The left indicates passion, evil, vice. The judge's sentence has to re-establish equilibrium in this asymmetric balance. In European popular traditions there still remains a negative judgement on left-handed people and on the use of the left hand, which is seen as the 'hand of the devil'.

The aesthetic category acts as a counterpoint to the moral category. Symmetry here is synonymous with harmony, and with the equilibrium of the parts, the forms, the colours. Beauty is derived from it. Polyclitus and Dürer, separated in time by almost two thousand years, both produced aesthetic systems based on the concept of proportion. The Greek temple and the Gothic cathedral were founded on symmetry and on aesthetic and religious expressions–even if the manner in one was more terrestrial and in the other more celestial. An analogous discourse can be developed in paintings, on sculpture, on the dance, the dramatic arts, etc., where the dynamism and the passionate nature of the asymmetrical forms contrast markedly with the serenity and harmony of the symmetrical ones.

The child comes into contact with all this through his parents, his brothers and sisters, his companions and the city or town where he lives. He internalizes at last these profound images, these original intuitions, through linguistic communication and visual experience. The maxims, affirmations, judgements, orders and aesthetic and moral evaluations comprise the stimuli coming from ambient, architectual or urban experiences, etc. They are inspired more or less directly by the very long cultural tradition the child inherits.

Another area in which symmetry plays an extraordinary role is the corporeal. The existence of a plan of vertical (but not horizontal) symmetry for the human body implies delicate problems of development, either in the formation of the bodily schema or in the process of lateralization. On the other hand, many traditional games of children properly point to the correct growth of one or the other. The child who tries to balance himself on a board, or on the edge of a garden strip, goes through a personal experience of balance and all that this evokes.

The visual representation of symmetry

In the scholastic activity of the earliest classes the concept of symmetry, in its most general form, is often used without the teachers themselves rendering an explicit account of it. This leads to incomprehension, and so to errors of interpretation and execution. A common example occurs in arithmetical operations. The use of the concept of equality and of the parallel concept of equivalence presupposes the conceptual category of symmetry. The evidence for this fact is found in every definition of equality and of equivalence (through the symmetrical property). Consequently, equality immediately brings to mind an image of stability and equilibrium. To facilitate this link, the image of a balance is often suggested. Indeed, a balance, with pegs on the arms for hanging weights, is often used to demonstrate equality. Besides, when arithmetical expressions are introduced, the operations of development tend, as a rule, to render evident the equality between the first and second member of the expression itself.

Nevertheless, the concept of an operation, in its most original, profound and intuitive form, is more akin to the general concept of movement, of transformation, of a passing from one situation to another (which itself is an event generated by some form of disequilibrium, such as the experiences of social life, of technical factors and of natural phenomena). Because of this, recent didactic opinion has recommended the introduction of a different symbolism, one that uses arrows instead of equal signs. So the tendency to base the concept of arithmetical operation on that of function is being widened rather than carried out in correspondence with logical operations. The same applies to the concept of the natural number, more opportunely set within the concept of a recursive function.

The failure to clarify the two aspects leads to many errors in the execution of even easy tasks of an arithmetical nature. Analogous considerations could be developed for the concept of geometrical equivalence and for the other connected concepts.

These brief indications already justify dedicating time and space to an adequate understanding and a valid representation of the concept of symmetry. In this case, the process of visualization retrieves the deep image of symmetry from the long-term memory of the student, (where it is still present in some way, even if confused) and raises it to the level of the short-term memory, that is of consciousness. This favours the rising in parallel of an opportune perceptive activity. The retrieval from the storing memory, and the perceptive activity, lead to an objectivization of the intuition of symmetry by means of a system of external signs. The process of geometrization, in this case, consists in providing the student with a system of valid and coherent representative signs. These signs, on the one hand, permit clarification and conceptual preciseness, and, on the other, foster a more or less immediate link with a section of mathematical theory of a geometrical type.

The visual representation of symmetry, using the design of a figure symmetrically arranged about an axis, can be progressively developed, starting from concrete experiences and activities. Nevertheless, we have to insist on the necessity of graphic schematization. Gonseth affirms that, in the graphic scheme, there exists the abstraction derived from the described situation, but that there also exists the concretization of that which the child has in his mind. The outline scheme has two aspects: one that is abstract in relation to the perceived reality, and one that is concrete in relation to what the student thinks. The visualization by means of a geometric scheme derives from the perceptual organization of the sensorial stimuli and from the recalling and utilizing of the images conserved in the long-term memory.

The visualization by means of a graphic scheme of a geometrical type (linked then without difficulty to a piece of mathematical theory) permits the analysis and development (by means of cognitive differentiation and definition, even verbal) of the properties of symmetrical figures. It also makes possible an understanding of the distinction between figures directly and inversely equal and of the introduction of the concept of symmetry as a geometric transformation of the plane. In addition, it allows the study of the composition of symmetries and the relevant properties.

The visualization, when correctly and significantly realized, becomes the foundation of a rich and articulated mathematical thought, which, however, does not remain isolated from the context of experience and of the study of the pupil. In fact, a successive, rational and coherent utilization of concepts connected with that of symmetry is fostered in this way—in, for example, the field of investigation relative to art. They become the basis of advancing and corroborating explanations of a physical, chemical and biological nature. But they also put forward adequate interpretations of pictorial works, architecture, sculpture, urban structures, and so on.

Conclusion

Mathematical education cannot underrate the process of visualization if it wants to ensure satisfactory progress in the knowledge and cognitive capacity of the student. It is the fundamental means of promoting the intellectual activity that mediates direct experience and permanent memory. The role of mental images is irreplaceable in every truly productive thought. Much psychological research, including the most recent, converges towards this conclusion. Moreover, representation by means of a system of graphic signs, coherent and contiguous with mathematical theory, forms a basis for a systematic analysis, for a rational control, for a definition and for a more precise utilization of interior intuitions. Without this bridge between the inner world and the real world, mathematical activity, which is above all characterized by an analytical and consequential way of thinking, encounters many difficulties.

Michael Mitchelmore

Spatial ability and geometry teaching in Jamaica

Introduction

Spatial ability is the ability to form and manipulate mental images of physical objects. This ability is called into play whenever one does geometry, since geometry is the study of the spatial properties of various figures abstracted from the concrete world of physical objects. Now geometry at secondary school, and at higher scholastic levels, also has an important logical component. So correlations between scores on achievement tests of geometry and on tests of spatial ability tend to be only moderate (Werdelin, 1961). On the other hand, primary-school children are still learning the basic visual concepts of geometry. So, at this level, one would expect to find a much closer relation between spatial ability and geometrical knowledge.

Very little cross-cultural research has been carried out on children's geometrical knowledge. It is the purpose of this chapter to look at some results obtained in one developing country, Jamaica, in order to form a picture of the sort of spatial difficulties that can arise as children learn geometry. Jamaica, although a developing country, is relatively advanced economically and has a long history of contact with Western culture.

Jamaican children's spatial ability

The clearest indication of the relative position of Jamaican children comes from a study I made a few years ago of the three-dimensional drawings and the reproductions of patterns of samples of pupils of comparable educational level and urban background in Jamaica, Ohio and England (Mitchelmore, 1980a). The rate of development between grade 1 and grade 9 was the same in all three samples, but the performance of the Jamaicans was different. Thus, for example, 62 per cent of the drawings of a cube made by Jamaican children in their fifth year of primary school (aged 11.5 years, on average) represented the cube in the simplest possible way—by a square. Only 6 per cent of the drawings made by English children at the same stage in their education (and, on average, 1.1 years younger) used this style. I recently gave the same tests to a sample of grade 5 students in the Federal Republic of Germany (average age 11.7 years). Their scores put them a further one-and-a-half years ahead of the English grade 5 children. This result emphasizes the difference between Jamaican children and children in these industrialized countries. But the unexpectedly wide variation among the three industrialized countries also suggests that the consequences of the difference need not be as serious as may have been thought.

Difficulties in drawing regular three-dimensional shapes can be attributed to an absence, in the child's concept of space, of an operational, Euclidean frame of reference, namely two perpendicular horizontal directions and a vertical direction. The problem is very apparent when young Jamaican children attempt to draw vertical telegraph poles along the side of a road in a picture (figure 1). In a recent study of typical Kingston schoolchildren (Mitchelmore, 1980b), I found that poles on the lower section were drawn within 10° of the vertical by only 12 per cent of grade 1 children. This proportion increased to 56 per cent by grade 5. Errors were greater on the upper section, and much greater on the middle section, presumably because of a desire to avoid overlapping the horizon on the road. As figure 1

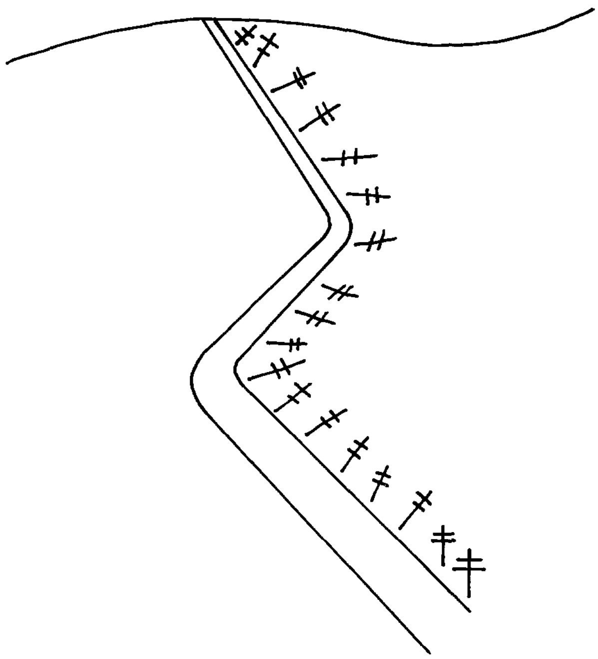
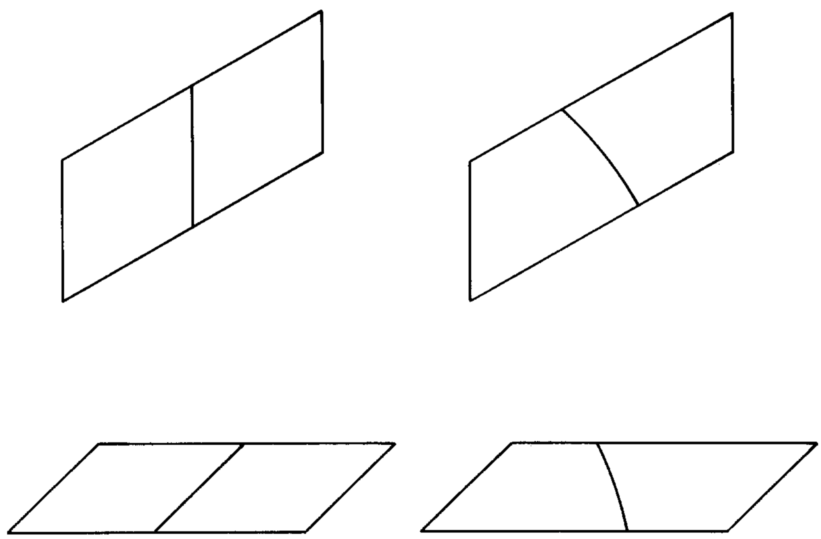


Fig. 1. Telegraph-poles task and completion typical of Jamaican grade 1 children.

shows, the tendency on all sections was to draw the poles perpendicular to the road. Similar difficulties were experienced in drawing the horizontal water level in sloping bottles; 62 per cent of the drawings made by grade 1 children showed the water surface within 10° of perpendicular to the sides of the bottle. Isaacs (1976) also found that grade 6 Jamaican children had great difficulty with this task, as they did on several other Piagetian conservation tasks.

My study (Mitchelmore, 1980b) also showed that the problem was not a simple one of a visualization of three dimensions or of physical knowledge about telegraph poles or water levels. Similar errors were made when copying the shorter median (the line joining the mid-points of the longer sides) of a parallelogram, as illustrated in figure 2. In 79 per cent of the copies made by grade 1 children, the line was nearer to the perpendicular than to the correct direction. The proportion fell to 44 per cent in grade 5, but it was not until grade 9 that more than 50 per cent of the drawings showed the median within 10° of the parallel to the shorter sides. Simpler figures were copied somewhat more accurately. Lines parallel to or perpendicular to a given line were copied within 5° in 59 per cent of the cases in grade 1 and 91 per cent in grade 5. Angles were copied within 10° in 40 per cent of the cases in grade 1 and 56 per cent in grade 5. Unfortunately, no comparable data on the copying accuracy in two dimensions of schoolchildren in other countries are yet available.



(a) (b)

Fig. 2. (a) Parallelograms with shorter medians, to be copied in adjacent congruent parallelograms. (b) Typical young Jamaican child's copies of the medians.

The impression gained from these results is that, in the course of their primary schooling, Jamaican children gradually develop the idea of direction, but that it remains somewhat general and intuitive, often leading to rather inaccurate drawings. Lines are drawn vaguely parallel to the given direction, but either children are not aware of the precise direction (except in the very simplest cases) or they do not accept that direction as an important criterion of an 'exact copy'. The task involving the median of a parallelogram illustrates children's difficulties quite clearly. Its successful completion appears to require that the child be consciously aware that the given line is parallel to the two shorter sides, so that the natural pull towards the perpendicular can be resisted; a vague sense of direction is insufficient. Readers can experience the problem for themselves by copying the lower parallelogram in figure 2 (a) quickly freehand; the median will probably be at least 10° out. The difference between adults and children is that adults would normally be much more careful when asked for an exact copy; children are apparently not able to exert such care. Poor drawings of regular three-dimensional objects seem to be an immediate corollary, since they require rather carefully drawn parallels or near-parallels.

Recent unpublished studies carried out by Jamaican student teachers suggest a concurrent lack of precision with shape, position and size. Two studies report difficulties in continuing the drawing of patterns in grades 3 and 9. Figure 3 shows a typical case; it looks as if the child started out correctly, then became impatient and completed the drawing with an arbitrary set of oblique lines. A similar thing happened when

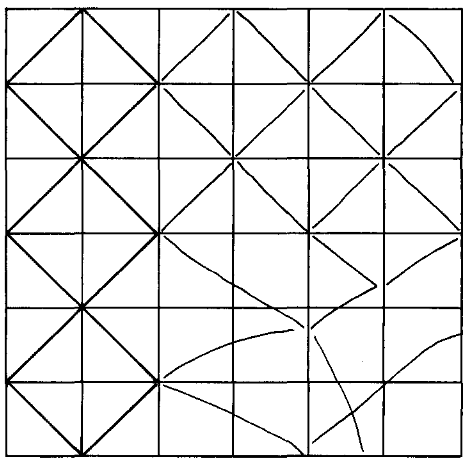


Fig. 3. Jamaican grade 3 child's attempt to continue the pattern given in the first two columns.

grade 6 children were asked to stick 2 cm squares on to 1 cm squared paper to make a tessellation. Many children stuck the first one or two rows on neatly, but then the tiles started to rotate slightly and overlap until they began to resemble fish scales. These examples suggest that the children did possess the necessary geometrical concepts, but were unable to apply them consistently in the face of the repetitive demands of the task. (As most investigators report that primary-school children very much enjoy such practical, pattern-making activities, it does not seem to be simply a matter of losing interest.)

Causes and consequences

Because of differences in physical environment, it may be reasonable to infer that some of the Jamaican children who were included in the sample may not have had a home environment similar to that of European children. Many homes in Jamaica lack special play equipment for children. They have fewer toys. The books are less sophisticated, and there is not as much variety in the television programmes for children to watch. Furthermore, such activities are often regarded as a waste of time and are discouraged in favour of household and other duties children are expected to perform. The effect of such a home environment is dramatized by the exceptions that come to light from time to time, such as the grade 4 son of a mechanic with a workshop at his house and the grade 6 boy who often helped his mason father; both boys did outstanding work on symmetry in rural classes consisting mostly of farmers' children.

The effect of an impoverished environment is often seen in more direct ways. Pupils may have difficulty doing such simple things as using a ruler to draw a line through two points. The difficulty in continuing patterns has already been noted; children may also not be able to colour them to make a regular pattern. (This could also have been another indication of the lack of precision referred to earlier.)

Children are further handicapped because very little geometry is taught in Jamaican primary schools. This is not surprising. It is only ten years since geometry was included in the syllabus of teachers' colleges. So the majority of teachers no doubt think of geometry as the secondary-school subject which they themselves found difficult because of the way it used to be taught to them, with the emphasis upon rigorous proof (usually of something that was obvious to the child), upon studies of the properties of shapes and upon accurate constructions. To such teachers the contemporary view of geometry and of how it should be taught will be quite alien, and one can sympathize with the five out of ten teachers who admit that they would need help before they could teach children with confidence.

The effect, however, on the teaching of geometry in the secondary school is predictable, and is confirmed by observation and in the reports of student teachers: so much time has to be spent teaching elementary concepts that only slow progress is made. The geometry syllabuses are similar to those used in comparable types of schools in the United Kingdom, so attempts to 'cover the syllabus' seem doomed to failure. Performance on geometry items in public examinations is always disappointing, but this could be the result of unrealistic expectations. To find out, I recently administered the same elementary geometry test (covering the names of the basic two- and three-dimensional shapes, some of their simpler properties, and basic directional concepts including angles) to typical grade 9 students in Jamaica and the Federal Republic of Germany. In the latter, the mean score among students destined to leave school after grade 10 was 74 per cent; those likely to continue beyond grade 10 had a mean of 85 per cent. The mean among the Jamaican high school students (who came from the top 20 per cent of the ability range and the upper half of the range of socio-economic status, and some of whom had been to private preparatory schools instead of the government primary schools) was 83 per cent. This suggests that these students had overcome whatever handicaps they had experienced at the beginning of secondary school. However, the mean among Jamaican grade 9 students from the non-selective secondary schools was much lower—just 50 per cent. Weighted averages provided overall estimates of 57 per cent in Jamaica and 79 per cent in the Federal Republic of Germany. The Jamaican average was below the mean score of 62 per cent achieved by the grade 5 students tested in the Federal Republic of Germany. In fact, it is estimated that the average Jamaican grade 9 student has learned about as much elementary geometry as the average grade 4 pupil from the Federal Republic of Germany. The fact that the gap is not as great as the difference in spatial ability may be due to the high verbal content of the geometry test. Despite the fact that English is a second language for the Creole speaker, Jamaicans always seem to perform much better in language subjects than in mathematics and science subjects.

A co-operative study of geometry teaching

It could be argued that children in Jamaica are comparatively so handicapped in spatial ability that geometry should not be taught, or at best should only be taught to older children. And, it might be said that however important geometrical-spatial-mechanical understanding may be to technological development, it may prove impossible to teach geometry to primary-school children. It was to explore further the effect of the teaching/learning situation on the learning of geometry in

Jamaican primary schools that I recently organized a group of twenty teacher interns to teach various geometrical topics to classes from grade 2 to 10 and to assess pupils' reactions to the experience, especially their learning of the ideas presented. The studies often fail to satisfy rigorous research criteria for objectivity, but, taken together, they prove conclusively that Jamaican children can learn geometrical ideas provided they are presented in the right way.

Twelve of the studies were undertaken at the primary-school level. One student taught a grade 2 class to recognize circles, triangles, squares and rectangles and to identify simple properties such as the number and the straightness of the sides. Another student taught a grade 3 class to recognize cubes, cuboids, cylinders and cones and to identify the shapes of their faces. Four students taught the ideas of parallel and perpendicular lines to classes from grade 4 to grade 6, and applied these ideas to the analysis of squares and rectangles. Two students dealt with repeating patterns in grades 3 and 6, bringing out the names and basic properties of various plane shapes. Three students taught ideas of mirror symmetry in grades 4, 5 and 6, reinforcing knowledge of the basic shapes at the same time as teaching concepts of distance and direction. One student taught angles to a grade 6 class, using the idea of rotation.

All the studies featured a very practical approach. Pupils were asked to look for examples of the ideas discussed both within the classroom and outside while on nature walks, and to bring examples from home or cut out of magazines. They sorted cardboard cut-outs, fitted cardboard strips end to end, folded and tore paper, made shapes with elastic bands on geoboards, constructed models from cardboard and made endless drawings. No sophisticated instruments were used. Where an instrument was needed, children made their own from card, string or tracing paper. Several students integrated their geometry teaching with arts and craft lessons, so that pupils had further practical experience of patterns and shapes. In most classrooms children worked in small groups on investigative or creative activities, and the results of group or individual efforts were displayed for all to see. All the student teachers reported that pupils became deeply absorbed in their work and often wanted to continue at the end of the lesson or unit. Several students noticed an increase in participation and co-operation among pupils (especially among those previously considered to be underachievers) and a decrease in disruptive behaviour. A number remarked on how much easier it was to give children mathematical discovery experience in geometry than in arithmetic lessons.

Test scores showed dramatic increases, typically from about 20 per cent to about 80 per cent as a result of a unit of twelve 30-minute lessons given over a four-week period. These are much greater gains than have been obtained in similar teaching studies in secondary schools.

They suggest that geometry may, in fact, be more easily learnt at the primary than at the secondary level. From anecdotal reports of performance on the pre-tests, it would seem that young children are well aware that their environment is filled with different shapes and configurations, but that most have never been given the opportunity for structured investigation or for learning the relevant vocabulary.

Incidentally, this co-operative study also shows that Jamaican teachers' colleges can turn out teachers who have mastered elementary geometrical concepts and who enjoy teaching the subject and are very creative at it.

Implications for primary-school education

It has been shown that young children in Jamaica readily grasp geometrical concepts if they are presented in an informed way. This implies that geometry should be given more emphasis in primary school than is the case at present, and that attention should be paid to the competence of teachers. Not only will later achievement in geometry improve because the basic ideas are more secure, but an eventual improvement in spatial ability can be expected. Both may facilitate industrial development. In fact, it could be argued that, in order to make up the huge backlog, elementary geometry should be given more emphasis than in developed countries. Certainly one unit a year is insufficient. Geometry can also be used effectively to teach certain desirable social skills.

While there is still a place for logical deduction as the child grows older, the geometry taught initially must be of a practical and exploratory nature. At this level, there is little need for formal geometrical instruments—they may even be a distraction—but there is need for adequate supplies of paper, card, string, scissors and such like. Scrap materials are often sufficient. There is certainly no place for 'chalk and talk'.

It is one thing to demonstrate that children can learn a geometrical topic, but quite another to show how learning can be sustained over the six or so years of primary-school education. Textbooks imported from the developed countries are unlikely to offer the teacher any guidance, since these assume a level of spatial development and experience that is several years ahead of the actual level of the pupils.

What is needed is a sequence of activities that will take the children from the stage of recognizing the overall shape of basic two- and three-dimensional figures to the point where they know their most important properties and the associated concepts, reinforcing knowledge at various points through visual-creative activities and leading to a fair degree of accuracy in geometrical representation. Without creative planning (which also implies testing), even practical geometry could be reduced to an aimless, repetitive and consequently dull and pointless activity.

It is an open question how existing primary-school teachers would accept a practical geometry programme. Certainly a great deal of in-service education would be required. There is much more hope for new teachers. If the teachers' college can lead students to analyse their environment geometrically and learn ideas in the same enjoyable and concrete way we believe they should teach their pupils, and if they can resist the temptation to place most emphasis on teaching students to form theorems or perform constructions, it would better serve the learning of basic geometrical concepts.

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Randall J. Souviney

Working story problems

In his chapter on conceptual difficulties associated with mathematics learning Ken Clements raises the important issue of the type of behaviour that should indicate mastery of a concept. He argues that children might not be considered to have mastered a particular concept until they can recognize exemplars of the concept in problem settings. In practical terms, he claims that it is insufficient simply to assess competence in elementary number facts and algorithms. Rather, what is important is the ability of students to recognize in a problem the operation(s) that will solve it.

This criterion of assessment seems to offer a practical example of the 'competence-versus-performance' dilemma faced by people involved in teaching and learning. Does a child's lack of performance infer an associated lack of competence? Such an approach to defining concept mastery, though somewhat more rigorous than that which obtains in common practice, could encourage teachers and researchers to focus more attention on the application of the skills of computation and measurement to appropriate problems in the environment. In this article, we will discuss some practical strategies that can be effectively used to help children to develop a facility for solving the class of problems commonly presented in textbooks as 'story' or 'word' problems.

Two types of problem

Problems can be classified into two general categories–story problems and process problems (Charles, 1981). Story problems are generally presented in a convergent manner. They require the student to read, with understanding, the written form of the problem and decide which operation(s) must be applied to the values given in the wording of the problem to solve it. Though this view of problem-solving is somewhat restrictive, it is widely used in schools and in textbooks, and it can provide an opportunity to develop necessary language skills.

Most realistic problems, however, are not easily isolated from their settings. The environmental setting of most real-world problems (i.e. those found outside the classroom), however, generally presents a mass of information and of variables that are extraneous to the solution of the problem at hand. A considerable effort of sifting relevant information from irrelevant information may be required before a clear understanding of the problem itself arises. To find an entry into such problems, an initial trial-and-error period may be necessary. General strategies, such as counting, sketching, estimating, measuring or seeking expert advice, are often useful in real-world problems.

Charles refers to such divergent problems as 'process' problems, since no single, clear algorithm is readily available, and more than one answer may be correct. Facility in solving problems of this type often makes it possible subsequently to tackle successfully a whole class of similar problems. Such skills are in great demand in any technological society. Further discussion of 'process' strategies is beyond the scope of this article. In what follows, we will focus attention on the first category of problems–story problems. For further discussion of 'process' problems, see Souviney (1981) and Charles and Lester (1982).

Language development and story problems

The ability to solve story problems successfully is constrained to a great extent by the student's level of comprehension of language. The results of a recent study of Papua New Guinea (Souviney, 1980) indicated a reduced reliance on visual memory between grades 2 and 6 among children who did well in mathematics and a commensurate increase in the relationship between skill in English (the language of instruction) and mathematical achievement.

Jones (1981) contrasted the rate of English-language learning as between first-language and second-language speakers. Using an information-processing model, he presents a convincing argument for his conclusion that the short-term memory needed to process key phrases in story problems exceeds the capabilities of many elementary-school children, particularly those who are learning in a second language. He offers an example of the following problem: 'The number 8 is 2 more than what number?' He found that a child who could retain only the word 'more' in mind while operating on the two numbers 8 and 2 would be likely to give the answer 8, since '8 is more than 2'. A child who could retain two words would be likely to give the answer 10, since '2 more' means 'add 2'. A child must be able to retain the three words 'is . . . more than' in order to be able to give the correct answer, 6.

Results of a study by Reed (1981) of tertiary-level mathematics students supports the existence of a relationship between the comprehension of language and the development of mathematics skills at the elementary (and early secondary) level. He defines three categories of language development: language comprehension, concept formation and mathematical symbolism. He argues that comprehension becomes increasingly important during the early stages of mathematics development. This dependence on, in this case the comprehension of English, diminishes at more advanced secondary and tertiary levels, giving way to the more demanding requirements of the parsimonious language of mathematical symbols.

Implications for teaching

If comprehension of the language of instruction is fundamental to mathematics learning at the elementary level, then the mathematics teacher has two choices: (a) to wait until competency in language is sufficient to support mathematics instruction, (b) to try to help the students to develop linguistic competency during the mathematics lesson.

The first alternative may be appropriate for young, native speakers who can be expected to develop the requisite language skills in a timely manner. For students learning in a second language, the first alternative may cause irreparable damage.

The second alternative follows Vygotsky (1962) and Feuerstein (1979). The assumption is made that expert assistance can greatly increase the range of tasks in which a novice can successfully engage. The accompanying operational principle involves providing external support to help students to accomplish a task or to solve a problem. Subsequently, this support is gradually withdrawn as individuals develop the required expertise. The process offers ample opportunity for success at each step. It allows the teacher to focus on specific elements of the task, and it provides strategies for corrective instruction if supports are withdrawn too rapidly.

Dynamic support for story problems

The following sequence offers an example of one possible set of dynamic support structures that enable the teacher to vary the short-term memory, the levels of written language comprehension, the numerical content and the complexity of the problem.

Developmental sequence

The manner in which story problems are presented can significantly affect the student's success in solving them. Consider the following categories of problem 'formats':

1. A story problem, with no numbers, presented orally, graphically or using symbols.

2. A number sentence used as the 'headline' to a story.
3. Writing an open-number sentence as the 'headline' to a given story problem.
4. Writing a number sentence, and computing its solution for a given story problem.

Under each of these headings several levels of external support are possible. The following examples may be instructive.

No-number problems. Try presenting problems orally, using pictures, or actually involving the children in concrete problem experiences. Children are often able to take in and process more key elements of a problem when it is presented orally, graphically or kinesthetically than they can from print. This is especially true of young children who are learning in a second language. For example, children might model early concepts in computation by 'adding and subtracting' numbers to and from groups of students of various sizes. Such 'people problems' provide concrete practice in relating instructions given orally to mathematical operations.

Initially, children have to identify only the operation called for in the story. This is the same as asking what the problem is telling us to do to find the answer.

Example 1. Venna and Hai bought some fruit at the store. What do they have to do to find out how much money they spent?

Answer: Add.

Example 2. Dogana has a fleet of trucks. Each truck has the same number of tyres. What can he do to find out quickly how many tyres he has in all?

Answer: Add or count and multiply.

Later, similar problems may be presented in written form. The students now have to determine the operation, and underline the word(s) or phrase(s) that indicate the operation to use.

Example 3. Anna needs to buy string to make a string-bag. She knows the *number of metres* of string on *one spool* and how much the *whole* spool costs. How can she find the *cost* of *one metre?* (Key phrases chosen may vary.)

Answer: Divide.

Make a story from its headline. Tell the students that they are going to write newspaper stories from 'headlines' you make up. Write a number-sentence 'headline' on the board, and ask children to make up a story problem to go with it. Children can draw pictures to tell the story as well.

Example 4. Headline: $3 \times 4 = 12$

Story: A canoe can hold 3 people. We have 4 canoes. As many as 12 people can go.

Open-sentence headlines are possible as well. Other children can solve the problems presented in such stories.

Example 5. We caught 5 fish. Then we caught 4 more. Three jumped out of the boat. How many fish did we have to eat?

Answer: Six.

Write headlines for story problems. Present stories in written form with the answer included in the problem. The students may have to write a number-sentence 'headline'. Drawing a sketch of the solution is an alternative way of 'writing' a number sentence.

Example 6. A truck can carry 750 kilograms of cargo. A bag of coffee weighs 50 kilograms. The truck can carry 15 bags of coffee.

Headline: $750 \div 50 = 15$.

Next present a problem leaving blank the places for the numbers. The students have to fill in the blanks and write the headline.

Example 7. Robert bought bunches of bananas. There were bananas in each bunch. He had bananas altogether.

Headline: $14 \times 10 = 140$ (for 14 bunches of 10 bananas).

Number sentences. Story problems can be presented that require the children to identify the appropriate operation(s), write an open-number sentence and compute its solution. With these it is not necessary for students to write their solution in a horizontal format. Vertical algorithms and mental computation should be encouraged. Again, start by leaving the number spaces blank, but end the story with a question.

Example 8. The school has round balls. There are children in the school. To be fair, how many children must share one ball?

Answer: For, say a school of 216 children having to share 11 balls, the children would write: $216 \div 11 = 19$, remainder 7, i.e., about 20 children.

The children should now be ready for story problems from the texts or on the board in which all the information is provided. Be careful to start with problems requiring only one operation for their solution, then gradually increase the difficulty to problems requiring two and three operations.

Example 9. (A two-step problem) Eddy had 30 cents, Anna had 35 cents and Richard had 25 cents. They wanted to buy a set of paints which cost 99 cents. Did they have enough money?

Solution: (a) $30 + 35 + 25 = 90$

(b) $90 < 99$. So they did not have enough money.

Finally, add information to the story that is not required for its solution, this gives children practice determining which facts are relevant to the problems at hand and which are not. The children should put an (x) through the numbers that are not needed. If the problems are presented in a text, a small marker may be placed over the unnecessary information.

Example 10. Alvin worked 5 hours painting 45 fence posts. Ella painted 50 fence posts in 4 hours. How many posts did they paint altogether?

Answer: 45 + 50 = 95 (the hours are unnecessary information).

Summary and conclusion

Adequate comprehension in the language of instruction has been found to contribute significantly to success in solving story problems. By carefully providing external support to the learner, it is possible to assist novices in solving problems. Systematically withdrawing the external support encourages students gradually to require less assistance in solving problems of a similar type.

The sequence of techniques for presenting story problems described above is not unique. Variables other than those that were present in the ten examples may be supported 'dynamically'. For example, social resources can be so tapped as to encourage persistence, self-confidence and brainstorming. One way is to establish a classroom environment that encourages problem-solving in groups of two to four children. Number facts can be supplied by allowing access to a calculator or to a two-entry table. This enables students with weak recall of the number bonds or those whose speed of computation is slow to participate successfully in problem-solving activities.

Problem-solving experiences that involve money, measurement, geometry and topics from the arts and social sciences should be included. Problems that relate to actual school activities such as class parties and community functions provide highly motivating opportunities for problem-solving. Such experiences offer children a rich source of settings where problems arise. This helps them to experience the wide range of realistic problems they are likely to encounter in life out of school.

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Peggy A. House and Thomas R. Post

Considerations for designing assessment programmes for teacher education

Prologue: a view of the status quo

Recent activities in mathematics education have resulted in calls in the United States from leaders in the profession for changes in various aspects of mathematics teaching (National Council of Teachers of Mathematics, 1980; Mathematical Association of America, 1978). A legitimate question in the face of such challenges is 'Change from what?' The answer can be found in large measure in three studies commissioned by the National Science Foundation (NSF) (Stake and Easley, 1978; Suydam and Osborne, 1977; Weiss, 1978). These studies demonstrated that the goals sought in the curriculum and the methodological reforms of the 1960s and early 1970s were not accomplished, and that mathematics education changed very little from 1955 to 1975. Gibney (1980) noted that the major findings that could be generalized from these three studies were:

A single textbook was the main source of content, with few other teaching materials requested or used.

In almost every class, the sequence of activities was to correct the homework, present selected examples and/or new material in a large group mode, then begin the new homework.

Little heuristic teaching was observed.

Non-instructional activities comprised a significant portion of the teacher's instructional time.

Standardized testing was in widespread use. A shift from norm-referenced to criterion-referenced tests was observed.

Achievement was found to be 'greater' in the elementary school, where a greater proportion of time was spent on developmental activities.

Vertical articulation of mathematical content between grades was generally lacking.

Few human resources were available to assist classroom teachers or to provide quality control.

Many teachers requested help in obtaining information about instructional materials and new teaching methods, including the use of hands-on manipulatives. Teachers saw other classroom teachers as the best source of new ideas. Local in-service training appeared to be more useful to elementary than to secondary school teachers.

In summarizing these and other conclusions drawn from the three NSF studies, Fey (1980) was led to conclude that the most discouraging feature of the three NSF studies is the consistent pattern of great differences between the apparent reality of mathematics education in most schools and the recommendations or practices of many prominent teachers, supervisors and professional organizations.

Clarifying the goal: teacher effectiveness

The note of disappointment conveyed by Fey's observation is shared by many in the mathematics education profession. If, indeed, such discrepancy exists between actual and desired teaching practices, then teacher education must be concerned not just with the training of teachers but with adjusting that training in a manner that will lesson the perceived gap. And since overall pedogogical effectiveness has long been used as a major criterion in evaluating teaching performance, it is reasonable to begin by attempting to characterize the effective teacher.

Research on teacher effectiveness

Research on teacher effectiveness to date has not provided a set of variables upon which to predict, with a reasonable degree of certainty, a particular individual's 'chances' of success in the classroom.

Studies have identified teacher characteristics that distinguish between experienced and inexperienced teachers. Ryans (1960) reported three such factors that later appeared in studies by other researchers: warmth, enthusiasm and business-like behaviour. Barr (1961) reviewed much of the research in teacher education since 1920 and suggested that many of the earlier attempts to identify and measure effective teachers were based on one of three types of information: (a) judgements by administrators, peers and students; (b) scores on tests of attributes presumed to be attributes of effective teachers, e.g. knowledge, attitudes, personality traits, college grades; and (c) student achievement. These procedures are still in widespread use, although instrumentation, analytical procedures and theoretical underpinnings are at present more sophisticated.

Begle (1979) cited a review by Morsh and Wilder of studies conducted between 1900 and 1952 in which the investigators criticized the wide variety of rating procedures and instruments for the non-comparability of results across time and location, and this may explain why such ratings have not correlated highly with student achievement. A fairly large number of studies has unsuccessfully attempted to correlate measures of teacher characteristics with measures of teacher effectiveness. This review pointed out that, of almost 700 correlations of this sort, fewer than 200 used pupil achievement as the criterion variable. In the measures of those correlations, a range of −.69 to +.81 was obtained, with an average of .065. A very small number of these was concerned specifically with mathematics. But, taken as a separate group, the results of those studies were similar.

Rosenshine (1971) summarized more recent research (after 1960), comparing teacher behaviour with student achievement. The report dealt with research in the three general types of teacher characteristics identified earlier by Ryans: (a) warm versus aloof, (b) systematic versus unplanned and (c) imaginative versus routine. Generally, teachers who had higher levels of the first of each of these pairs of characteristics were more effective with students, although the results were not conclusive. Other studies, dealing with ratings of teacher knowledge, aptitude, attitude, experience and preparation, were likewise non-conclusive.

Some studies (Rosenshine, 1971; and the National Longitudinal Study of Mathematical Abilities, NLSMA) have established quite conclusively that teacher effectiveness varies over time *within* specific individuals. Geeslin (1972) correlated measures of effectiveness of two groups of teachers over a two-year time span, and obtained indices between .01 and .35. Only 12 per cent of the variance in the second-year scores was accounted for by the first-year scores.

NLSMA also examined student achievement as it related to thirteen variables in teacher background, using large numbers of students across age, location and various levels of cognitive ability. Extensive univariate and multivariate analyses were conducted. Of a possible 2,704 main effects, only 530 (20 per cent) were statistically significant. Begle noted that the results 'do not support the claims that any of these characteristics are powerful indicators of teacher effectiveness. Even the strongest, Credits in Mathematics, had a significant positive relationship to mathematics achievement only 24 per cent of the time' (Begle, 1979, p. 43). Similar NLSMA results were obtained for seven variables in teacher attitudes. For teacher attitudes, only 24 per cent of the main effects (348 out of 1,456) were statistically significant.

These, and other data relating to teacher effectiveness, led Begle (1979, p. 54) to take a rather negative view of the future of research in this area. He wrote:

> Probably the most important generalization which can be drawn from this body of information is that many of our common beliefs about teachers are false, or at the very best rest on shaky foundations. Thus, for example, there are no experts who can distinguish the effective from the ineffective teacher merely on the basis of easily observable teacher characteristics . . . . Consequently we have to admit that we do not at present know of any way of selecting, in advance, the effective teachers or of knowing whether a particular teacher preparation program does indeed produce effective teachers.

The disappointing nature of the research on teacher effectiveness is undoubtedly due to the fact that student outcomes are the result of complex interactions among teacher, students, subject matter, instructional setting, socio-economic status and, no doubt, other as yet unidentified variables. To identify those variables which do contribute to student learning and the relative degree to which they do so is a task yet unaccomplished. The use of multivariate statistical procedures, only recently in widespread use, holds promise in identifying and describing clusters of characteristics associated with the effective teacher. At present, however, it does not seem feasible to use criteria of teacher effectiveness as the primary model for the evaluation of teachers, whether pre-service or in-service.

Domains of teacher performance

Smith (1980, p. 7) suggested that pedagogical knowledge evolves from an attempt to match principles of effective teaching (not effective teachers) to the domains of performance of classroom teachers. In contrast to previous approaches to teacher evaluation, such an approach would greatly de-emphasize the importance of characteristics of individual teachers as supposed indicators of effectiveness. It would focus instead on the ability of an individual to translate a well-defined body of pedagogical knowledge into instructional 'moves' in the teaching situation. It must be pointed out that Smith believed that a thorough training in the various disciplines was an essential pre-requisite to serious pedagogical studies. An obvious task necessary to the implementation of such an approach is the delination of these 'principles of effective teaching'.

Such an approach would, in effect, reverse the more popular procedure of beginning with objectives and then defining the activities and procedures appropriate to their realization. Smith argued that the 'objectives first' approach is justified only when the field of knowledge is well established and understood. As noted in the previous section, this is not the case with teacher effectiveness.

In considering what is known about teaching, we find that teaching has been conceptualized in two ways: didactics (direct teaching) and heuristics (indirect teaching). Didactics is the art and science of precise instruction geared to imparting specific knowledge and skills. Heuristics is the art and science of teaching students to inquire, to discover, to search and to find out for themselves. Both are appropriate in the modern classroom, although personal styles will differ from teacher to teacher.

Smith (1980) observed that knowledge about these two modes of teaching falls into six categories. These constitute the domains of teacher performance alluded to above. The categories are:

1. Observation.
2. Diagnosis of: (a) student abilities; (b) learning obstacles; (c) environmental conditions; (d) programmes of instruction.
3. Planning of: (a) short-range programme; (b) long-range programme.
4. Management of: (a) space, time and resources; (b) instruction; (c) students.
5. Communication with: (a) peers; (b) parents and other laymen; (c) students.
6. Evaluation of: (a) student achievement and conduct; (b) instructional programme.

These categories provide a useful framework within which to embed appropriate evaluation criteria. A later section of this chapter will provide examples of how this can be done.

Others have conceptualized the act of teaching in ways that have implications for teacher evaluation. Cooney (1980, 1981) made a study of how teachers make decisions and observed that 'teachers gather and encode information, generate alternatives, and select a course of action' (Cooney, 1981, p. 68). Three broad categories were identified: (a) cognitive decisions, relating to content and the selection of teaching method; (b) affective decisions, relating to inter-personal aspects of teaching; and (c) managerial decisions, relating to time allocation and the overall co-ordination of the classroom environment. Within each of these, there are, of course, many subcategories. But this conceptualization broadly paralleled that suggested by Smith, wherein the domains of teacher performance were initially identified and the specific objectives subsequently emerged from them. Cooney also extended Henderson's (1963) view of teaching as a three-part relation, made up of sequences of teacher actions, the subject matter and the behaviours of those taught, to include a fourth consideration: the setting. This scheme may be a convenient one, but it appears to be oversimplified in the light of research into these four areas, when a more complex view emerges.

Current thinking tends to see the teacher as a dynamic, information-processing being, who must simultaneously process and modify parameters within each of the four categories. It is not surprising, then, that earlier attempts to correlate single or small groups of teacher characteristics with student achievement did not lead to a definitive characterization of teacher effectiveness.

Cooney (1980) supported professional judgement as 'a viable and desirable means of assessing outcomes of a teacher education programme . . . given the state of the art of both teaching and teacher education'. This, perhaps, seems to be a disappointing statement in the light of the vast amount of research that has been done. But it is certainly an understandable one given the complexity of the task. In spite of this, Cooney went on to advocate including basic techniques and concepts in the teacher education programme, while acknowledging at the same time that successful teaching will always be greater than the sum of its parts. To illustrate his thesis, he drew an analogy with the champion tennis player who is more than a person who has mastered the basic backhand, forehand and serve. Even though the fundamentals of teaching are not precisely understood, 'the best of what we know' should be included in any professional programme. We agree with this. Unfortunately, it is by no means clear what body of pedagogical knowledge is important enough to be included in teacher education programmes, and, by implication, to be evaluated in the classroom setting. This situation clouds the issue for persons seeking to develop a coherent programme of teacher evaluation.

Where, then, are we when one attempts to assess validly and reliably the teaching effectiveness of pre-service mathematics teachers in instructional settings? How can research be used to guide these attempts? Are there guiding principles to assist with the task? We turn now to the consideration of some of these issues.

Developing effective teachers

Although research on teacher effectiveness has produced little specific information on what, exactly, constitutes an effective teacher, it is generally accepted that teacher effectiveness is a key component (if not *the* key component) in student learning. Further, studies of teacher effectiveness generally support the conclusion that it is what the teacher does, not what he or she is, that determines effectiveness as measured by pupil learning. Thus, however ill-defined the concept of teacher effectiveness, a primary goal remains to increase the effectiveness of teachers.

Medley (1979, p. 11) concluded that there are two important ways to improve teacher effectiveness: 'One is by improving the way teachers are evaluated, and the other is by changing the way teachers are educated.' It is hard to imagine that either approach would be truly productive without the other because the ultimate goals of both evaluation and education are to increase the success of the individual.

Setting goals for programmes

What, then, do we expect from a teacher education programme? Peck and Tucker (1973, p. 943), reviewing research on teacher education, suggested several generalizations based on their review. Among them were the following:

1. A 'systems' approach to teacher education substantially improves its effectiveness. Such an approach consists of a series of steps which recur in cyclical fashion. They are (a) precise specification of

the behaviour which is the objective of the learning experience; (b) carefully planned training procedures aimed explicitly at those objectives; (c) measurement of results in terms of the objectives; (d) feedback to the learner and the instructor; (e) re-entry into the training procedure; (f) measurement of results again following repeated training.
2. Teacher educators should practise what they preach. When teachers are treated as they are supposed to treat pupils, they are more likely to adopt the desired style.
3. Direct involvement in the role to be learned produces the desired teaching behaviour more effectively than remote or abstract experiences such as lectures on instructional theory.
4. Using any or all of the techniques mentioned, it is possible to induce a more self-initiated, self-directed, effective pattern of learning not only in teachers, but, through them, in their pupils.

If, following the recommendations above, a systems approach is adopted, the necessary first step is to specify the teaching behaviours to be developed. While it is not within the scope of this discussion to suggest specific teaching behaviours, we assume that those behaviours will, by and large, be subsumed in domains of performance similar to those identified by Smith and discussed earlier in this chapter, namely: diagnosis (including observation), planning, instruction, management, communication and evaluation. Examples will be given later.

Further, we recognize that teacher performance is a complex interaction not only among behaviours in the various domains of performance, but also at various levels of performance. These levels are best described by adopting a taxonomy of teaching behaviours as described below. This taxonomy (House, in press) is deliberately developed to reflect Bloom's (1956) taxonomy, and the names for the categories are chosen for that reason, although more appropriate names may be found. The six levels are the following:

Knowledge

The level of knowledge is concerned with facts, processes, theories, techniques and methodology related to instruction. It also includes knowledge of mathematics and of the curriculum and materials of school mathematics. In the taxonomy of teacher competencies, this domain subsumes all of the levels of Bloom's *cognitive* taxonomy. It is the component of teacher education usually associated with the college classroom. It is usually measured by paper and pencil, or by other conventional classroom methods.

Comprehension

The level of comprehension is concerned with the performance of selected behaviours under controlled conditions, such as peer teaching, micro-teaching, simulations, role playing, etc. It is a demonstration that the individual can do something, and the behaviour to be demonstrated is usually called for in an explicit manner, so that the individual is conscious of the goal of demonstrating the desired behaviour.

Application

The level of application refers to planning and administering learning activities and materials in a classroom setting. It is evidence not only that the individual can do, but that he/she does do. Application involves the use of appropriate teaching skills at the proper time, or with the desired frequency, as a part of the normal teaching style.

Analysis

At the level of analysis, the teacher responds to pupil, to subject matter and environmental cues, to select, organize and administer effective programmes and lessons. The teacher recognizes the constituent elements of the curriculum and the relationships among them and sees them as an organized whole. The teacher also responds spontaneously to students as individuals and his/her actions and decisions flow from a consistent and conscious rationale.

Synthesis

At the level of synthesis, the individual orchestrates his/her teaching behaviour into a personalized whole, interiorizing and professionalizing the teaching skills and combining the underlying competencies into an effective style unique to the individual.

Evaluation

At the level of evaluation, the teacher judges the effectiveness of his/her teaching according to various internal and external criteria, including pupil progress toward desired goals, and he/she modifies the teaching in the direction of greater effectiveness.

Combining this taxonomy of teaching behaviours with the domains of teaching performance identified earlier, we can form the Teacher Performance Matrix represented in figure 1. This matrix enables teacher educators to clarify and specify activities directed towards the dual goals of improving both evaluation and teacher education as will be illustrated in a later section. Examples of material for the lettered cells are given in Table 1.

Figure 1. Teacher-performance matrix

Levels of teaching behaviour	Domains of teaching performance: Diagnosis	Planning	Instruction	Management	Communication	Evaluation
Evaluation						(F)
Synthesis			(E)			
Analysis				(D)		
Application		(C)				
Comprehension					(B)	
Knowledge	(A)					

Advantages of the matrix approach

Several advantages accrue from the design of the matrix described here. One is that it focuses attention on aspects of teaching not readily described in specific behavioural objectives. More global, less atomistic competencies are suggested. From these are derived instructional goals and alternatives. Another advantage is that it directs attention to the continuing growth and development of the teacher and to relate pre-service and in-service educational goals. For example, programme developers and evaluators can distinguish those performances that may be expected of beginners (many of them will be performances at the first three taxonomic levels, although some demonstrations of higher-order performance are to be expected) from those performances that may be more appropriately expected of experienced teachers. Such differentiation should result in more realistic expectations for beginning teachers, and may help to alleviate some of the frustrations commonly experienced by beginning teachers. It should also help teachers to evaluate their own behaviour and to plan for their own continuing professional growth.

The taxonomy also encourages us to perceive the development of competencies from one level to another. As an example, we might consider behaviours related to the teacher's use of evaluation in teaching. At the level of knowledge, these include understandings about different types of tests; the reliability and validity characteristics of tests; the uses of tests for formative, summative or diagnostic purposes, etc. At the level of comprehension, competencies might include demonstrations of the teacher's ability to construct tests that are appropriate for certain specified purposes. The application of these competencies indicates that the teacher actually does employ various types of evaluation. At the higher levels of analysis, synthesis and evaluation, it becomes apparent that the teacher's evaluation instruments and processes reflect the objectives of his/her instruction, and that the teacher uses them to improve instruction and more effectively to help individual pupils.

The affective characteristics of the teacher also constitute a strand that may run through all levels of the taxonomy. For example, affective competencies at the level of knowledge may include knowledge of or recognition of the defence mechanisms or approach/avoidance techniques of pupils, or knowledge of the importance of teacher support through empathy, warmth or a positive attitude to pupil learning. At the level of comprehension, the teacher may be expected to demonstrate behaviours he or she has learnt for giving reinforcement or to identify clues to pupil or teacher attitudes in simulated situations. At the level of application and beyond, teacher competencies may include the spontaneous use of reinforcements, empathy, etc. in teaching situations. Affective goals may also include positive changes in teacher attitudes toward the various components, personnel or systems in education, and a willingness to modify his or her teaching behaviour as he or she develops new competencies.

Finally, because the performance matrix is based on the behaviour of teachers, a variety of methods of assessment is available. Performances at the level of knowledge are generally assessed by conventional classroom methods, including paper-and-pencil tests. Both pre-service and in-service teachers can be expected to demonstrate their knowledge about essential mathematics; about relevant theories, techniques or processes of education, or about other content deemed appropriate. They will do this by written and/or oral responses including tests, papers, reports, discussions and other forms of presentation.

Performances at the level of comprehension may be assessed through controlled situations for a limited time, such as peer teaching, microteaching, simulations or role playing. Among the means of assessing these are observations by teacher training personnel or other experts, rating scales, checklists and video-tapes. These apply at both the pre-service and the in-service levels.

At other levels, performance normally presumes demonstration in actual classroom settings over an extended period of time. These may apply more appropriately to assessments of in-service needs and, to a more limited degree, to student teachers. The means of assessment include all those identified for comprehension. They may also include the assessment of pupil learning as one indicator of teacher effectiveness or need.

Selected examples

While the cells of the matrix in figure 1 are neither clearly delineated nor disjoint, it is generally possible to locate desired goals in teacher education in a primary position within the array. To illustrate elements of such a programme for teachers of secondary-school mathematics, Table 1 elaborates representative competencies, objectives, learning activities and assessments corresponding to the six lettered cells in figure 1. These examples lead to the final consideration: evaluating teacher-education students.

Table 1. Selected examples of material for the lettered squares in Figure 1

Performance goals	Instructional objectives	Learning activities	Means to assess
Cell A			
The student will recognize the importance of diagnostic teaching in mathematics.	For a given mathematical topic, identify the component concepts and prerequisite knowledge and skills. For a given mathematical topic, identify probable sources of pupil difficulty or error. Complete a task analysis for the given topic. Suggest appropriate instructional activities for developing the specified topic. Suggest appropriate remedial activities corresponding to anticipated or identified pupil errors.	Formal instruction on learning and developmental theories. Student completion of task analysis for selected topics. Discussion of common sources of pupil error. Analysis of pupil work to identify error patterns. Independent student search for resources and instructional alternatives appropriate for remediation of identified errors. Case study and/or tutoring pupils with learning difficulty. Examination of available diagnostic instruments.	Tests to assess knowledge about learning, diagnostic teaching and other concepts. Oral and written reports from student diagnosing a particular pupil's difficulty. Critique of learning activities planned for remediation. Critique of outputs (task analysis, error patterns, test interpretation, etc.)

Table 1. Selected examples of material for the lettered squares in Figure 1 – *cont.*

Performance goals	Instructional objectives	Learning activities	Means to assess
		Administration of diagnostic test of pupils and analysis of pupil performance. Observation and tutoring in special education mathematics classes.	
Cell B The student will demonstrate effective communication skills.	Demonstrate the ability to implement a variety of communication patterns in the classroom (e.g. lecture, guided discovery, discussion). Recognize defensive strategies and nonverbal behaviours of pupils. Use open-ended questions in instruction. Initiate and sustain discussion. Use pupil questions and comments in developing instruction. Give feedback and positive reinforcement to pupils.	Discussion of the effects of various forms of communication in the classroom. Classroom observation and systematic recording of communication patterns. Application of interaction analysis in demonstration lessons or in classroom observations or in video tape of student's own lesson. Identification of levels of questions in demonstration lessons. Planning of questioning sequence for selected lesson. Comparison of demonstration lessons taught according to lecture, guided discovery, discussion or other approach. Observation of classroom situations for examples of pupil non-verbal behaviour.	Written or oral discussion of the effect of communication in the classroom. Direct observation or video tape of lesson. Analysis of lesson plans. Interaction analysis performed on students.

Performance goals	Instructional objectives	Learning activities	Means to assess
Cell C The student will plan and teach a unit of instruction.	Plan appropriate unit and lesson objectives. Develop lessons related to the overall unit objectives. Provide appropriate pacing, practice, application and feedback during instruction. Vary the learning activities within and between lessons. Manage the classroom in a manner that promotes pupil learning. Evaluate pupil progress and learning outcomes.	Development of written unit and lesson plans. Independent search of resources to identify useful lesson strategies and materials. Teaching unit to secondary school pupils. Development of pre-tests, post-tests and quizzes for unit. Measurement of pupil learning during and after the unit. Revision of lessons after instruction.	Analysis of written plans by supervisor. Observation by supervising teacher and evaluator. Classroom observation by peers. Video tape of lessons for self-evaluation. Discussion of lessons with supervisor. Measurement of pupil achievement of unit objectives.
Cell D The student will see pupils as individuals and respond accordingly.	Give individual recognition and acceptance to pupils. Listen to pupils. Expect success of every pupil and take measures to bring it about. Differentiate assignments according to pupil needs. Involve pupils in planning units and activities. Adjust instruction to meet individual pupil needs. Recognize variations in pupil learning styles and identify differences in ways in which individual pupils approach problems.	Formal instruction on relevant psychological theories; individual differences; needs and characteristics of slow learners, gifted, handicapped, etc. Practical experiences in special education programmes. Practical experiences tutoring gifted pupils and pupils with learning difficulties. Administration of diagnostic tests, Piagetian tasks, etc. to pupils. Systematic observation of selected pupils in a variety of settings in and out of school. Case studies and discussions of these with experts and with other educators. Video taping lessons for review and discussion by student, peers and/ or instructor.	Observation of student interacting with pupils. Checklists to assess frequency of student's use of positive reinforcement, feedback to pupils, using pupil comments or questions in instruction, etc. Examination of lesson plans and teaching logs to determine extent to which lessons and assignments are differentiated. Pupil feedback and evaluation of instruction.

Table 1. Selected examples of material for the lettered squares in Figure 1 – *cont.*

Performance goals	Instructional objectives	Learning activities	Means to assess
Cell E			
The student will combine the underlying competencies into an effective, personal teaching style.	Model the mathematical behaviour desired of learners. Display enthusiasm for mathematics, for teaching, for learning and for pupils. Recognize the mathematical aspects of situations and integrate mathematics with other areas of learning. Regularly relate classroom learning to past learning, future topics and pupil interests and experiences. Respond to classroom contingencies in an appropriate and consistent manner. Help pupils take ownership of their own learning tasks and assist them in bringing closure to those learning tasks.	Reading of professional journals and participation in professional meetings. Engaging in problem solving on a regular basis. Building files of teaching resources, strategies, materials. Discussions with persons who regularly use mathematics in various contexts. Interviewing or polling pupils to determine their attitudes and interests. Discussions with teachers of other subjects on the relationships between mathematics and those disciplines.	Observations in classroom over time. Comparisons of observations to assess consistency of behaviours. Pupil feedback. Pupil growth and learning. Peer evaluation of the teacher's contribution in meetings, workshops, etc.
Cell F			
The student will judge the effectiveness of one's own teaching.	Display confidence in one's own ability to teach mathematics. Evaluate one's own teaching in terms of internal criteria (e.g. accuracy and logical consistency), external criteria (e.g. appropriateness for the given learners), and pupil progress toward goals.	Formal instruction and independent study of relevant research in mathematics education. Interacting with colleagues in professional meetings, workshops, seminars. Conducting formative and summative evaluations of pupil progress.	Observations by colleagues or outside evaluators. Self-evaluation. Pupil feedback. Measures of pupil growth and learning. Peer review of profesional contributions.

Performance goals	Instructional objectives	Learning activities	Means to assess
	Identify teacher behaviours that inhibit pupil learning and propose modifications to those behaviours. Evaluate curricula and materials for appropriateness in furthering one's instructional goals. Plan and evaluate lessons in the light of current relevant research.	Observing colleagues and visiting other schools. Preparing immediate and long term proposals for one's own professional development.	

Evaluation in the teacher-education programme

Establishing the criteria of performance and assessing the attainment of students in them are major obstacles in any programme. The lack of sound data, as mentioned earlier, on the relationship between the performance of the teacher and the learning of the pupil means that the identification of behavioural criteria is based on subjective judgement. These criteria must flow from the specified objectives. These, in turn, should represent one's best professional judgement, based on the available, albeit fragmentary and tentative, knowledge. They are defensible, not because they are based on certain knowledge, but because they represent a systematic attempt to ensure that nothing essential is overlooked, and because they lead to assessment that is specified and descriptive and that affords precise, corrective feedback.

In establishing the behavioural criteria, it is common to consider six types of criteria:

1. *Knowledge* that the student is expected to acquire.
2. *Outputs* (products, events) that the student is expected to produce.
3. *Behaviours* that the student will demonstrate.
4. *Attitudes* that the student will display.
5. *Consequences* (usually pupil learning) that result from the student's intervention.
6. *Experiences* that the student will have.

This last category, that of experiences, acknowledges the fact that not all outcomes can be specified in advance. An example of an 'experience' would be one where the student will have had practice in a variety of grade levels, mathematics courses, ability levels and patterns of school organization. The outcomes of such experiences are not predictable. They might, however, include the student's recognition of the uniqueness of individuals and of situations, or a clearer insight into one's own strengths, weaknesses, attitudes and preferences.

There are several approaches to setting these behavioural criteria so as to include specifying the expected frequency of behaviours, identifying the expected degree of accuracy or adherence to some standard, establishing a rating system and an expected performance norm, or assessing the result of the student's performances in terms of their pupils' learning and growth. In any case, the criteria must take account of such questions as: How is the behaviour to be demonstrated? When? How often? In what settings? In how many settings? Under what conditions? With what level of proficiency?

As a rule, the assessment of student competencies serves two major functions: a descriptive function that facilitates the recording and analysing of behaviour in order to provide corrective feedback and a judgemental function that equips programme personnel and students to make go/no-go decisions. Although the problems associated with assessment are many and complex, the more explicit the behavioural criteria are, the easier they are to assess and evaluate. Also, the more specific and useful will be the feedback that flows from the evaluation. Ultimately, one is concerned with the stability and growth of competencies. Hence, a one-time demonstration that the student can do something is far less useful than an on-going evaluation of behavioural patterns made at appropriate times and with appropriate frequency. A constant concern is that the evaluation should adequately sample the student's behaviour so that it can yield defensible inferences of competence.

There are many means to hand for assessing student knowledge and performance. The varied means of assessment suggested in Table 1 for the six representative competencies and their related enabling activities illustrate the range of possibilities. The appropriateness of the chosen means is related to the type of behavioural criteria that are being assessed. In general, criteria of knowledge are assessed by all the conventional classroom means (quizzes, examinations, papers, problems solved, discussions, oral presentations, etc.). Outputs are examined and evaluated according to specified norms or characteristics. Behaviours can be counted and/or rated. Attitudes are elicited directly from the student's written or oral communication, or are inferred from his or her non-verbal communication. Consequences are evaluated by measuring pupil growth. Experiences are counted, logged, described and/or submitted to self-evaluation by the student.

The worksheet below presents a sample rating scale that illustrates some of the considerations that enter into evaluating the teaching performance of a student teacher. It is presented here as an example of the range of behaviours to be considered. (This worksheet was developed for use in supervision of student teaching at the University of Minnesota.)

Student teaching performance
Supervisor's worksheet

Student Teacher:

Supervisor:

How would you rate this student teacher's performance of these skills?

0. No opportunity to observe
1. Very poor performance
2. Poor performance
3. Fair performance
4. Good performance
5. Very good performance

Planning

1. Constructing lesson plans.	0	1	2	3	4	5
2. Planning units of study.	0	1	2	3	4	5
3. Justifying instructional plans.	0	1	2	3	4	5
4. Stating explicit goals and objectives.	0	1	2	3	4	5
5. Selecting a model of teaching to fit an instructional goal (e.g. inquiry model).	0	1	2	3	4	5
6. Selecting specific teaching procedures to help students attain objectives.	0	1	2	3	4	5
7. Finding instructional materials needed to implement plans.	0	1	2	3	4	5
8. Selecting instructional materials according to specific criteria (e.g. reading level).	0	1	2	3	4	5
9. Selecting instructional methods that match student developmental needs.	0	1	2	3	4	5
10. Designing learning experiences to help all students make the most of their abilities.	0	1	2	3	4	5
11. Taking account of special needs (e.g. handicapped).	0	1	2	3	4	5
12. Sequencing learning experiences to accomplish objectives.	0	1	2	3	4	5
13. Designing the classroom environment to promote learning.	0	1	2	3	4	5
14. Building motivators into instructional plans (e.g. variety).	0	1	2	3	4	5
15. Evaluating an instructional plan according to explicit standards.	0	1	2	3	4	5
16. Predicting possible effects of instructional plans on students.	0	1	2	3	4	5
17. Preparing outlines for effective oral presentations.	0	1	2	3	4	5

Instructing

1.	Using computers in the classroom.	0	1	2	3	4	5
2.	Using audiovisual equipment in the classroom.	0	1	2	3	4	5
3.	Using an effective presentation style.	0	1	2	3	4	5
4.	Asking different kinds of questions to stimulate different kinds of student learning.	0	1	2	3	4	5
5.	Giving ongoing feedback to guide student performance.	0	1	2	3	4	5
6.	Modifying plans to deal with unexpected events of student responses (e.g., an interest in pursuing a particular topic).	0	1	2	3	4	5
7.	Demonstrating principles or procedures.	0	1	2	3	4	5
8.	Explaining how or why something happened.	0	1	2	3	4	5
9.	Defining concepts.	0	1	2	3	4	5
10.	Improving students' questioning and discussion skills.	0	1	2	3	4	5

Managing the classroom

1.	Providing rewards to motivate students.	0	1	2	3	4	5
2.	Encouraging self-direction in students.	0	1	2	3	4	5
3.	Dealing with behaviour problems during class.	0	1	2	3	4	5
4.	Clarifying expectations for student behaviour.	0	1	2	3	4	5
5.	Holding students responsible for their actions.	0	1	2	3	4	5
6.	Resolving interpersonal conflicts.	0	1	2	3	4	5
7.	Interacting with students who have special needs.	0	1	2	3	4	5

Diagnosing

1.	Keeping anecdotal records of a typical student behaviours.	0	1	2	3	4	5
2.	Using a variety of methods to determine student status.	0	1	2	3	4	5
3.	Integrating information from several sources to determine student needs.	0	1	2	3	4	5
4.	Determining student developmental level.	0	1	2	3	4	5
5.	Participating in case conferences with staff.	0	1	2	3	4	5
6.	Identifying non-instructional factors which may be limiting student progress.	0	1	2	3	4	5

Evaluating

1.	Establishing evaluation criteria.	0	1	2	3	4	5
2.	Interpreting standardized achievement tests.	0	1	2	3	4	5
3.	Constructing tests to measure student progress.	0	1	2	3	4	5
4.	Using criterion-referenced tests.	0	1	2	3	4	5
5.	Evaluating personal teaching effectiveness.	0	1	2	3	4	5
6.	Critiquing student performance (e.g. strengths, areas which need improvement).	0	1	2	3	4	5
7.	Using evaluation results to improve instruction.	0	1	2	3	4	5
8.	Reporting information about students to parents.	0	1	2	3	4	5
9.	Assigning student grades.	0	1	2	3	4	5
10.	Evaluating the ethical consequences of one's own actions.	0	1	2	3	4	5

It is obvious that the type of evaluation described above represents a departure from the common notion of research, where it is traditional to set high criteria for 'certainty' ($p < .01$, for example), the implication being that it is better to reject something that is true than to accept something that is false. Here, in contrast, evaluation is thought of as an attempt to establish descriptive data about what the student can or cannot do. In this, the intent of the evaluator should be to provide precise, and specific, feedback so that the student will know not only how he or she is evaluated, but also why the evaluation is undertaken and what should be the focus of future learning activities. No teacher is helped by an occasional ten-minute visit from a supervisor, or by an evaluation that simply states the lesson was adequate, or that the pupils were not learning. No one is more concerned about classroom problems than the teacher, who must spend every day in that classroom, and most teachers sincerely want to deal with their problems and to continue to improve their teaching. They need specific advice that recognizes their strengths and that offers ideas that can lead to action in building on those strengths while attending to their weaknesses.

Conclusion

It is one thing to recommend adopting a systematic approach to teacher education through goals specified in advance; it is quite another to determine what those goals should be. It is one thing to expect that learning activities should be related to objectives, and that evaluation should reflect demonstrable competence; it is quite another to establish how these objectives are to be achieved. Obviously, the responsibilities of educators in making decisions are enormous.

While programmes may share many common characteristics, each, in the end, is unique. Needs, personnel and situations differ. It is just not possible either to develop the definitive programme or to transplant a programme unchanged from one setting to another. What can be done, though, is to outline major steps to be taken, important questions to be asked and answered, significant decisions to be made. It is hoped that the considerations offered in this chapter for designing assessment programmes for teacher education will contribute to the professional dialogue that aims ultimately at improved effectiveness for all teachers.

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Andrew Herriot

Contemporary styles in pre-service teacher education for primary-school mathematics

Primary mathematics teacher education in southern Africa

PRISM '81, a seminar held in Lesotho on primary-school mathematics for southern Africa, devoted considerable attention to teacher education. The issues raised and the recommendations made included teachers' needing more prescriptive and detailed guidance, extending the training period to three years, raising the entry qualification to Ordinary (O) level, merging method and content in an integrated course related to the environment and providing teachers with a basic kit of materials, including all varieties of hardware and software.

In practice, the training of a primary mathematics teacher varies considerably from country to country in respect of the duration of the course (in years), the qualifications for entry, the amount of teaching practice given, the instructional materials available to the teacher in training and the type of course. These differences are clearly brought out in Table 1.

National Teacher Training College, Lesotho

The Mathematics Department of the National Teacher Training College (NTTC) formulated and developed a mathematics programme for the training of all pre-service primary-school mathematics teachers in Lesotho (International Seminar on Mathematics Education, Swaziland, 1979). Two distinct three-year programmes are offered, leading to two distinct certificates:

The Primary Teacher's Certificate (PTC) for Junior Certificate (JC) students who will qualify to teach all subjects in the primary school.

The Advanced Primary Teacher's Certificate (APTC) for O-level students who will qualify for responsible positions in the primary school either in an administrative or a supervisory capacity.

One of the innovative highlights of both programmes is the internship

Table 1. The training of teachers of primary mathematics in certain countries of southern Africa.

Items compared	Mauritius	Malawi	Zambia	Swaziland	Lesotho
Entry qualifications	O-level (Cambridge)	J.C.[1] –2yr M.C.E.[2] –4yr	O-level (Cambridge)	J.C.[1] –2yr	J.C.[1] –3yr
Teaching practice	16 weeks	3 hours per week	10 weeks	12 weeks	40 weeks
Instructional materials used	Teachers' guides, textbooks	Topic units	Textbooks, handbooks, activity cards	Teachers' guides, textbooks	Teachers' guides, workbooks, self-instructional material
Type of course	Content and method separate	Content and method separate	Content and method separate	Content and method integrated	Content and method integrated

1. Junior Certificate.
2. Malawi Certificate of Education.

year, the second year of study. All students are placed in schools throughout the country, and, under supervision, practise skills and implement ideas learned on the campus.

After six years in operation, the college is beginning to make an impact on the state of primary-school mathematics teaching in Lesotho. Table 2 gives the actual intake and output figures for both programmes. Apparent discrepancies in APTC output arise from the fact that, in certain cases, experienced and mature APTC students were excused the internship year.

Table 2. Candidates for PTC and APTC, 1975 to 1981: numbers entering each year of the course and numbers graduating

	PTC				APTC			
Year	I	II	III	Graduates	I	II	III	Graduates
1975	37	–	–	–	12	–	–	–
1976	155	33	–	–	36	12	–	–
1977	180	141	32	29	36	23	24	26
1978	186	174	134	122	39	–	58	54
1979	221	165	168	161	14	34	2	2
1980	242	198	181	175	32	11	37	36
1981	263	235	203		50	28	14	

The college also provides in-service training for primary-school mathematics teachers through a 'credit system', whereby both untrained and trained teachers are upgraded in knowledge and in skills as well as in status (NTTC, 1982).

Programme content

The aim of the PTC programme is to produce primary-school teachers capable of teaching mathematics at any stage in the curriculum. The APTC programme covers the same material as the PTC programme, while aiming at producing teachers capable of leadership in primary-school mathematics, either as headteachers or as staff members with supervisory responsibilities within the profession. Four courses are offered. They are designated MA 1.1, MA 1.3, MA 10.1 and MA 10.3. Their descriptions are outlined here:

MA 1.1. Primary Curriculum Studies (early years). The course aims to introduce students to those topics that are taught in the early years of the primary school. Emphasis is placed on teaching techniques, including the preparation of lessons, materials, worksheets, apparatus and visual aids. Importance is also given to the improvement of mathematics content. Topics include sets, operations for whole numbers, fractions and decimals, shape, measurement and graphic representation.

MA 1.3. Primary Curriculum Studies (later years). The course aims to complete the study of those topics taught in the later years of the primary school. Further emphasis is placed on the preparation of suitable materials for teaching. Mathematics content is developed. Topics include investigations, how children learn mathematics, strategies for teaching, making worksheets, measurement, applicable arithmetic, introduction to curriculum development and language in primary mathematics.

MA 10.1. The course supplements MA 1.1 and is given to those students specializing in primary-school administration (APTC). Topics include those of MA 1.1, together with principles of learning, sequencing of mathematics content, planning a topic, writing objectives, lesson planning, model lessons and teaching techniques.

MA 10.3. This course supplements MA 1.3 and is given to those students who continue to specialize in primary-school administration (APTC). Topics include those of MA 1.3, together with significant figures, metric units, number patterns, ratio, rate, proportion, scales, symmetry, area and volume, circle and sphere, investigations and graphical illustration, basic statistics, writing teaching materials, testing techniques and study of textbooks and other resource material.

Table 3 gives the structure of the two programmes in primary-school mathematics.

Table 3. How PTC and APTC are made up

	Year 1	Year 2	Year 3
PTC	MA 1.1 3 hours per week	Internship 30 weeks	MA 1.3 3 hours per week
APTC	MA 1.1 3 hours per week MA 10.1 3 hours per week in third term[1] only	Internship 40 weeks	MA 1.3 3 hours per week MA 10.3 3 hours per week in second term[1] only

1. Each academic year comprises three 'terms' of eleven weeks each.

The methods used for teaching all components in the primary-school mathematics programmes include: half-hour large-scale lectures (two per week), one-hour small group (twenty-five students) activity sessions (two per week), self-instructional materials (Mathews, 1978), teachers' guides on specific topics and micro-teaching sessions.

Large-scale lectures

Each week, a topic or an idea is developed that follows closely the outline given in the current primary-school mathematics syllabus of the Ministry of Education. Where possible, a multi-media approach is used when presenting the material. The whole year-group is therefore exposed, in a well-prepared lecture, to clear objectives, concrete materials (home-made and commercially structured) and appropriate activities. Great care is taken in making maximum use of the large-scale lecture; information can be transmitted in an organized way, allowing the key ideas to be further pursued in a more informal setting. Tables 4 and 5 show how the lectures develop throughout the two-year programmes.

Small-group activity sessions

During these hourly sessions, students have the opportunity to engage in making apparatus, visual aids, games, etc.; carrying out planned activities at two levels, their own and that of the children; using concrete materials prepared in class; referring to resource materials such as the primary-school syllabus, textbooks, guides and materials produced in the college; practising content questions taken from the item bank; discussion with the tutor of the next assignment; and reviewing the content of the large-scale lecture, particularly from the point of view of the objectives of the course.

Table 4. The sequence of topics discussed during the large-scale lectures of the first-year mathematics course MA1

Week	Term one	Term two	Term three
1	Sorting	Length Arbitrary units (Money) Metric units	Fractions: Multiplication
2	Relations: comparison ordering	Mass (Money) SI[1] units	Fractions: Division
3	One-to-one correspondence Cardinal number Sets[1]	Subtraction algorithm	Building prisms
4	Ordinal number Conservation	Multiplication algorithm	Capacity, Volume Volume[1]
5	Addition Subtraction Ordered pairs[1]	Division algorithm	Areas Metric units Units[1] of area
6	Multiplication Division	Symmetry 2-D	Decimals: place value addition
7	Classifying 2-D, 3-D shapes	Tesselation Areas (arbitrary units)	Decimals: subtraction multiplication
8	Pictorial representation	Angle of a polygon Angle[1] Degrees	Decimals: division Bicimals[1] Tricimals
9	Place value (Dienes blocks) Ancient number systems[1]	Fractions Concept	Time
10	Place value (abacus)	Fractions Addition Subtraction	
11	Addition algorithm (money)		

1. Background lecture

Table 5. The sequence of topics discussed during the large-scale lectures of the third year mathematics course MA 1.3

Week	Term one	Term two	Term three
1	Area and perimeter investigations (include mm, g, ml) SI[1] revision	Ratio, Rate	Reading scales
2	Further investigations	Proportion	Simple surveying
3	How children learn mathematics	Percentages	Scale drawing
4	The making of a worksheet	Profit and loss	Equations Simple balance
5	Approximations (significant figures, decimal places)	Civic arithmetic	Directed numbers: addition
6	Factors, Multiples, Primes	Civic arithmetic	Directed numbers: subtraction
7	Number patterns	π	Curriculum development
8	Time (24-hour clock)	Areas of 2-D figures	Curriculum development
9	Speed, Time, Distance (Pictorial)	Volumes of prisms	Language problems
10	Properties of triangles and quadrilaterals	Surface area	
11	More 3-D shapes (classifying, building)		

1. Background lecture

The small-group session is the time for individual work, group activities and small-scale lectures. Its form will depend very much on how the group reacts to the instructional materials. If, for example, the session involves making apparatus, students will, on instructions from their tutor, prepare the essential concrete materials before attending that particular class. Since the lecture is given during the week before the small-group sessions, and as students receive their instructional materials at the beginning of each term, it is easy to plan activities for small groups well in advance.

Activity workbooks

Workbooks, covering the content of the syllabus of the college, have developed within the department of mathematics. During each week a new topic is introduced according to a predetermined sequence, as shown in Tables 4 and 5. These are compiled and presented to students in the form of a termly workbook which is published by the college. The content of each workbook is based on a format consisting of five sections, some of which may not be appropriate in any particular week. The five are: a summary of the lectures, background mathematics that may be relevant, practical activities suitable for both the children and the student, item bank questions and instructions for an assignment (four per term). The workbooks are designed to cover all topics contained in the primary-school mathematics syllabus of the Ministry of Education. They can be used in a self-instructional situation or as a resource both on and off campus. They are very useful during the intern-ship year as they offer ideas for lesson planning and classroom activities.

Self-instructional materials (SIM)

Single-topic units have been developed, and they provide the student with material that is supplementary to the basic programme. SIMs have been written on sets, sorting, relations, one-one correspondence, bar charts and circle graphs. SIMs that are not written in isolation but as part of the learning experiences are valuable. They provide students with another useful resource. Copies of all SIMs are placed in the library. Multiple copies are printed for use in the activity sessions whenever appropriate.

The development of the skills of writing SIMs continues. It has been enhanced by the creation of the Instructional Materials Resource Centre (IMRC), a separate department within the Ministry of Education. IMRC will assist and advise on such matters as the design, the development, the editing and the evaluation of a SIM. The policy of the college is that SIMs should be written as a co-operative effort and that ideas for further topics can and should be obtained from students and teachers.

Teacher's guides

This material arose out of the need to explain in greater detail certain topics within the syllabus. Topics such as number facts, fractions and mathematical games have been covered. The main emphasis here is on a detailed explanation of the development of mathematics in the primary school, referring to objectives, activities and materials. The various stages at which a topic should be introduced are presented in the form of a teacher's guide that gives a student of primary-school mathematics a clear perspective of the curriculum. In preparing teachers for their internship year, they should be made aware of sequencing in mathematics. The production of useful teacher's guides is an on-going departmental activity.

Micro-teaching

Each student during the first year has an opportunity to practise, and to master certain teaching skills using micro-teaching techniques. These provide practical experience with children, immediate feedback using a video-tape recorder, and consultation with and the advice of an experienced tutor. Micro-teaching is co-ordinated by the professional studies department, and all subject departments are required to contribute. Contributions may take the form of student assistance when preparing instructional objectives, suitable concrete material and help with examples of probing questions which may be used during activities. Since the activity sessions allow for flexibility in approach, it is often possible on these occasions to discuss problems which arise in micro-teaching sessions.

Resources

An integral part of the training of a primary-school mathematics teacher is the development of an ability to produce suitable classroom materials, particularly charts and visual aids, various items of apparatus and worksheets and evaluation items.

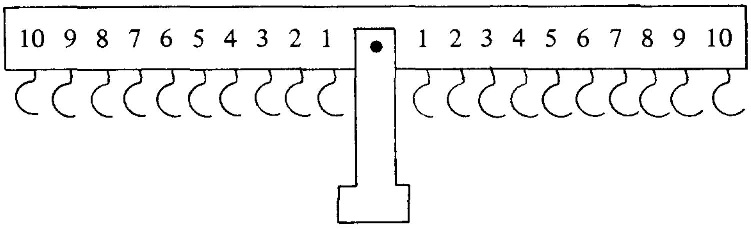
Assignments

All students are accordingly required to prepare a kit of simple apparatus, to include such items as a spike abacus, models of the number system in base 10, fraction boards, a clock face, a geoboard, various three-dimentional shapes, an angle indicator, a trundle wheel, various calibrated containers, a clinometer and a simple balance. An example of an assignment given to students on the preparation of a piece of apparatus is given below.

Assignment: to construct an equalizer.

1. What is the point of this assignment:
   This piece of apparatus can be used for various levels in the primary school and therefore will be very useful for the teacher as well as the pupil.
2. What do you already know about this piece of apparatus?
   You have investigated the use of the equalizer in class.

3. What will you do?
   You should make an equalizer that has 10 hooks on either side.
   It must balance well.
   You should attach 10 rings to your equalizer.



   You are free to create your own design, but it will look something like the one shown above.
4. What materials will you need?
   Wood or masonite. Nails or hooks of some kind. Rings or bottle tops with holes in them.
5. What will you hand in, and how will it be assessed?
   One equalizer. The equalizer must balance accurately with and without rings (10 marks). It should be strong and long lasting (5 marks). It should be neat and well made (5 marks). Total: 20 marks.

Charts and visual aids that students prepare as part of their kit include those used for teaching symmetry, percentages, one-one correspondences, number patterns, fractions, shapes and scales of various types. Students are shown during lecturing sessions how to use a visual aid in a mathematics lesson. A great deal of importance is attached to good visual material that can be used in the classroom to create initial interest and to assist in the formation of new ideas in mathematics. An example of an assignment requiring students to prepare a chart is given below.

Assignment: to prepare a chart.
1. You are going to prepare a large visual aid which can be used when teaching either a large group or a small group.
2. The chart will have geometric diagrams and will show, by shading, a wide variety of fractions (including decimal fractions) and their equivalent percentages.
3. You should make use of the following geometric shapes: square, rectangle, circle, triangle, pentagon, hexagon, octagon.
4. Your chart will illustrate the following percentages:
   2½, 15, 25, 30, 33½, 45, 60, 62.5, 75, 90.

Worksheets

In primary school, worksheets have become, in recent years, one of the more suitable alternatives to a textbook. They usually consist of a specific number of instructions or questions that either aid further understanding or act as an instrument of evaluation. Students are encouraged to match evaluation with their objectives; practising writing worksheet questions is a very important educational exercise which allows them to test whether children have or have not understood a specific idea.

Students are also shown how a question or an instruction on a worksheet can lead to an activity that may eventually require a discovery. Guidelines are laid down, as the example from the weekly workbook shows here, offering a model for students in the first instance to emulate. The fact that worksheets are given both as assignments and evaluation questions in examinations demonstrates the importance that the department of mathematics attaches to skill in writing them. An example of guidelines for making a worksheet follows:

Design

Simplicity—ask one question or give one instruction at a time. Do not make them too difficult.

What is the 2 equal to in the number 204?

Activities—students can be asked to do something related to teaching, make a drawing or work out a calculation.

Estimate the length of the classroom in metres.
Measure the length of the classroom.
Draw an angle of 60°.
Divide 10.2 by 10

Order

Simple to more difficult—the questions should progress and increase slightly in difficulty.

$$\frac{1}{3} + \frac{2}{3} =$$

$$\frac{1}{3} + \frac{1}{6} =$$

$$\frac{1}{3} + \frac{1}{4} =$$

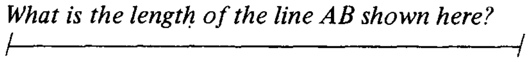
Teaching order—the questions should as far as possible follow the order in which the topic was taught as in the above set of fraction questions.

Format

Space for the answer—all answers or diagrams should be written next to each question and therefore sufficient space should be left.

Make a picture of the number 382 using the given abacus here.

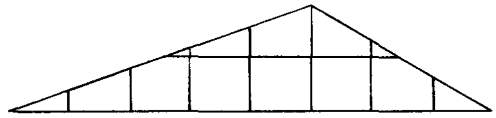
What is the length of the line AB shown here?



A B

Style
Diagrams and pictures–the value of a worksheet is increased with the addition of interesting questions.

What is the area of the triangle, shown here, in cm^2?



Language
Use easy words and short sentences:
'what is', 'draw', 'work out', etc.

Internship

This is a one-year teaching experience taken in the second year. During this year, students assume a normal teaching load in a school and are expected, under supervision, to apply the professional skills and knowledge acquired in the first year on campus. Preparation for this year includes practising writing suitable objectives, planning lessons, making resource material, becoming familiar with the syllabus and current textbooks and matching evaluation items with objectives.

The supervision takes the form of weekly visits from a tutor who is attached to the area where the schools are cited. The supervisor, who has been specially trained in a programme offered by the National University of Lesotho and supported by Unesco, advises approximately twelve student teachers. The assessment of teaching has been the subject of much debate since the inception of the internship year. Current practices are clearly summarized in the report of a seminar sponsored by Unesco in 1978 on Evaluation in the Education of Teachers. This seminar took place in Maseru and was organized by the College Evaluation Committee. The conclusions and recommendations that emerged from the group involved in discussing teaching competencies were as follows:

Thirty competencies were identified under the following major headings: (a) organization, prior to the lesson (planning) and during the lesson; (b) techniques (methods and materials); and (c) relationships.

Objective evaluation is impossible, but subjective evaluation may be unfair. A list of competencies (acting as a check-list) will aid both the supervisor and the student in training to assess progress and ability.

There are only three possible internship grades: fail, for those students for whom it is clear that they have chosen the wrong profession; pass, for the majority of students in training; excellent, for those exceptional students who are recognized by both supervisors and tutors as teachers of quality.

Supervision visits are made by college tutors on a regular basis. The visits are combined with workshops for the interns, the in-service students and teachers in the area. Workshop topics include mathematical games, making apparatus, measurement and place value and operations.

All workshops given by college tutors emphasize a practical approach. So teachers, at the end of all one-day workshops, acquire materials and ideas for use in their own classroom. This educational activity is considered by the department of mathematics to be a vital service to all teachers. The valuable feedback obtained during these weekly sessions enables college tutors to sustain a crucial link with the primary school. Plans are under way to complete the construction of thirty-five resource centres, each situated at one of the 'sites' where NTTC students follow their internship year. Each centre will be equipped with a duplicator, a filing cabinet, classroom materials, books, paper, SIMs, teacher's guides, workbooks and other materials that have been produced by educational centres such as the National Curriculum Development Centre. These centres are intended as meeting places where teachers can discuss educational matters and share ideas concerning the teaching of mathematics.

Assessment

Two methods are used to monitor performance and progress on campus:

End-of-term examinations consisting of objective items (Hamilton, 1976 and 1977), and essay questions so structured as to lend themselves to objective marking when linked with a carefully prepared marking schedule.

Continuous assessment, comprising various assignments of a practical and written nature.

All new scores are collected, averaged and adjusted, using both weighting and standardizing techniques (Herriot, 1978). A minimum level of performance is required before a student is allowed to progress to another year. Students who do not reach the minimum level of competence are allowed to retake an examination. Criterion-referenced techniques are used when assessing the performance of such students (Mathews, 1980).

Conclusion

The innovative features of seeming relevance in the training of primary-school mathematics teachers at NTTC are: the activity work-books, designed with a broad structure, but with sufficient flexibility for use by various tutors and teachers; the varied methods of learning, such as micro-teaching and self-instructional techniques; the internship year, combining workshop visits; and the preparation of a kit of materials.

NTTC is about to embark on an evaluation of the internship year. As a result of this major exercise, essential information will become available about the innovations that have been introduced in the training of primary-school mathematics teachers. Feedback from students and teachers does suggest that the concept of a detailed workbook, such as those developed here at NTTC, should be explored further so that all teachers, including those in training, may benefit from them.

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Alan Osborne and James Schultz

Contemporary styles in in-service education for primary-school mathematics

Through the ages, the pursuit of intellectual growth and the continual acquisition of additional skills have been the hallmarks of good teachers. Many institutions and agencies regard the greatest value of in-service education as the provision of help and support to teachers in their search for growth and pedagogical skills.

The approaches and the mechanisms that facilitate in-service education are many and varied. Some indication of the range of sponsoring institutions and the scope of the programmes they provide are given by these examples:

A local school system offers a workshop for teachers on instructional materials used in teaching mathematics. The workshop is conducted by a teacher who has used the materials.

A local university helps teachers, first, to acquire those insights that will identify mathematically talented youngsters, and second, to develop the special materials needed for instructing them.

A business enterprise provides financial support for teachers who are making a study of mathematical applications, and arranges visits to the local plant to demonstrate the uses of mathematics by the employees.

A regional education authority helps teachers and principals to see the advantages of adopting a new syllabus in mathematics and to anticipate the problems which could arise from so doing.

The participation of institutions and agencies in in-service education through such programmes is increasing throughout the world. It is now less common to find individuals who consider the training of teachers to be 'finished' with the end of pre-service training. Education agencies, school systems, professional groups, universities and business groups are accepting (and sharing) responsibility for extending the education of teachers beyond their initial training.

During the last few decades, the various approaches to in-service education have so evolved that it is now possible to identify the principles likely to assure success. The purpose of this chapter is to describe them. And specific recommendations are made towards the constructive use of teachers' time and the use of other resources in designing in-service education for the improved learning of mathematics by children.

There are two basic tenets of in-service education. The first is this: the in-service programme should be closely related to the mathematics being taught. The second is this: the in-service programme should seek to augment leadership at the local level. Teachers are best served if those responsible for the in-service programme have a sense of the goals and objectives of the instructional programme in mathematics and of how well they are being realized. In most instances, the resources available for in-service education are so limited that the sponsors can ill afford to spend money on activities that are marginal to the mathematics taught in school.

The first of the tenets mentioned above does not restrict the level of the mathematics discussed to the levels taught in school. Indeed, it is essential that teachers should be at ease with mathematical ideas at a considerably higher level than that at which they are teaching. Neither is it implied that in-service education should be aimed solely at the implementation of existing curricula. The thorough graph of an existing programme is but a first step toward a goal of greater merit: preparing teachers to participate in the development of new and better programmes. This is particularly true in developing countries, where quite possibly there are no existing programmes that meet the specific cultural needs of the children. Here change can best be brought about by strong local leadership.

Evidence is accumulating that one of the most powerful reasons for ensuring a strong relationship between the school mathematics programme and the in-service programme is that, when this is done, teachers are more highly motivated to participate in and to contribute to the activities of the in-service programme. Surveys of teachers and supervisors in the United States (Osborne and Bowling, 1977) indicate that basing in-service education on topics of concern to teachers appeals to the teachers' sense of professionalism. This appeal is enhanced if the teachers participate in planning the in-service activities and in assessing the needs of those likely to attend. Teachers display a sense of responsibility for the programme that results in their working harder to ensure its success.

Assessing needs

It is crucial to the success of in-service programmes to be able to anticipate the difficulties of teachers who will take part. Not only does an accurate assessment assure a well-focused effort and the constructive use of resources, it also ensures a close correlation between the in-service training and the school mathematics programme. Processes that are sensitive to the perceptions of teachers generate an enthusiasm for the in-service activities. It is, for example, likely that conscientious teachers, who are sensitive to situations that arise in the classroom, will seek help to deal with the needs that emerge from their teaching. Some of these needs may arise from a difficulty in answering particular questions which students will have asked, or they may concern the complexities of ministering to the special needs of slow learners or gifted children.

The identification of in-service needs is not easy. Teachers who try to do so may be handicapped by a perspective narrowly limited to their own classroom. They may need more information before they can make sound judgements about the success of the school programme in mathematics. Supervisors or administrators within the school system may be more aware of the different in-service needs since they will have access to more information about the performance of children in mathematics. University teachers or government officials who are tuned to current trends and who may have a broader vision of a country's or region's needs may see the role of in-service education from a still different vantage point. The ability of people who are not primary-school teachers to assess the needs of primary-school teachers depends on both the amount of contact that they have with the schools and their awareness of national needs.

Some practices that have assisted communication between classroom teachers and those who are involved in assessing the needs for in-service programme include classroom observations, preliminary meetings, pre-course questionnaires and post-course evaluations.

Several types of classroom observations have been found useful in assessing needs. Examples include:

University faculty members visiting the classroom and noting strengths and weaknesses of instruction.

Classroom teachers observing university faculty members or observing head teachers in a 'master class'.

Observing demonstration teaching.

Classroom teachers observing one another.

Preliminary meetings between teachers and those responsible for developing the in-service programmes provide useful information for in-service leaders. Such meetings also bring home to the teachers the fact that a genuine attempt is being made to ascertain their needs with a view to incorporating discussion of them into the in-service course. Pre-course questionnaires provide another efficient source of information on desirable content. They may also alert the in-service leader to any discrepancies between what the teachers expect and what is being planned. Discrepancies so discovered can either be resolved or justified. Post-course evaluations help to determine the content of future in-service courses.

The assessment processes described above are effective. Yet they have the advantage of being quick and inexpensive. Other relevant procedures can be less so. For instance, it is appropriate to examine factual evidence of how well students perform in mathematics in subsequent grades, in test performances and the like, particularly if the tests are related to the aims of the curriculum. When teachers participate in the assessment of needs, their judgement of the effectiveness of the curriculum is better received if it is tempered by evidence of their students' performance. However, in seeking evidence from students' performance, planners must be reasonable in the amount and type of information sought. It can be expensive in terms of the advantage gained.

Types of courses

In-service programmes can assume many forms. Commonly used forms include the following: workshops, university on-campus courses, university school-site courses, outside consultants, professional meetings and professional readings.

Workshops are especially fruitful if they provide a high degree of involvement in minimal time. Workshops should be held at a time and in a place that are convenient to the teachers. As they call for considerable preparation, workshops generally cater for only a small number of participants. They differ from traditional courses in that teachers work with materials and engage in sundry activities rather than attend lectures.

University courses held on the campus are convenient for the faculty but are often inconvenient for the teacher participants. A particular inconvenience is the red tape that seems endemic to most modern universities (registration, fees, parking and transport). When arranging courses on campus, planners must give every consideration to the inconvenience facing the participants. Such courses are typically designed to meet the needs of participants from a number of schools. Since the needs in one school are seldom identical with those in another, the course is usually designed for an average need that may not cater completely for all. Not every school, of course, is conveniently located for university-based course work. For teachers in remote rural schools, special arrangements must be made for in-service courses on university campuses.

There is a growing trend for university mathematics educators to run courses on school sites. Such courses save time and expense for teacher participants. Moreover, it is possible for the in-service instructor to arrive early enough to visit classrooms and to work directly with the children in a demonstration teaching mode. Experience shows that on-site courses are popular with teachers, and the instructor gains credibility with teachers, particularly if he takes part in classroom activities and gives demonstrations.

Meetings of professional associations and the literature they send out to members provide another form of in-service education. The help they give is less direct than that of a formal course of training, and it is up to the teacher to take the step of making the connection between what is communicated at the meeting and the local school situation. Membership of a professional organization usually covers a subscription to one or more journals, as well as giving access to books and publications for teachers. Agencies which support in-service training can achieve a high return on money spent for membership in professional organizations. Some organizations encourage institutional membership. Thus a member school can make available to all the teachers the journals and other print materials in the professional library.

Issues

Several major issues confront those responsible for in-service programmes. Among these, the following appear to be significant in most settings:

What priorities should be given to the various in-service needs of teachers?

How can school systems, universities, teacher-training institutions and governmental agencies help to meet the in-service needs of primary teachers?

How should the problems caused by cultural differences that arise when establishing in-service programmes for developing countries be dealt with?

These issues are not independent. Rather, they are highly interrelated. In what follows, each issue will be discussed with a view to providing a rationale for specific recommendations.

What priorities should be given to the various in-service needs of teachers?

To the teachers themselves, their needs often seem immediate: 'What can I do in class tomorrow?' But, as those responsible for mathematical education are aware, an understanding of mathematics and of its pedagogy go hand in hand as coherent bodies of knowledge. There are discrepancies between the immediate need and the ultimate goal. They can lead to difficulties in in-service education.

Long-term needs are especially important for primary-school mathematics teachers. One of the most common deficiencies of primary-school teachers is their tendency to see mathematics as a collection of rigid algorithms or as a 'bag of tricks'. If in-service education concentrates exclusively on what is to be taught tomorrow, it will fail to form the judgement that is needed in assessing new instructional approaches or different curricular content. The goal for in-service education should be to build sufficient judgement and expertise that the in-service course will not need to be repeated in the immediate future. Short-term needs should not, however, be ignored. Giving attention to them has two advantages: what is learnt can be applied immediately, and concern for them helps to generate interest. The latter can serve the cause of the long-term goals.

Meaningful courses for teachers require a balanced, harmonious blend of content and method, incorporating both theory and practice. The report of the National Advisory Committee on Mathematical Education (NACOME, 1977) in the United States recommends that dichotomies should be avoided in setting policies for mathematics education. The report highlights the pitfalls associated with concentrating exclusively on such polarities as either content or method and either theory or practice. Most primary-school teachers grasp the content better when it is presented within the context of their day-to-day work with children. The teaching of pedagogy is more effective when attention is paid to the mathematical ideas that are the objects of instruction. To divorce discussion of content from that of method is to fail to recognize the nature of the decisions that primary-school teachers must make. Only rarely does a teacher find it possible to make a decision that is exclusively mathematical or uniquely methodological.

The incorporation of both theory and practice into in-service education is necessitated on two grounds. First, primary-school teachers must come to feel that their investment of time and energy in the in-service programme has paid off. If the course is purely theoretical, so that making the connections to classroom practice is left to the teachers, the 'homework' is dramatically increased. Most primary-school teachers find working with children so taxing that they have to weigh carefully the gains from time devoted to in-service training against time devoted to children. Productive teachers will consider their responsibilities to the children to be their first priority, and will show little patience with in-service courses that are exclusively theoretical. Second, the learning of theory or of practice is enhanced by experience with the other. A sound theory provides a conceptual foundation that assures that practices are more readily learned and remembered by the teacher. Practice serves a role in learning about theory that is analogous to the role of working examples in the learning of mathematics.

Workshops, presentations and other in-service activities can be organized around themes such as 'How can we best teach fractions?' or 'What can be taught using Cuisenaire rods?' We claim that, of these two examples, the former theme is to be preferred. This is because we teach mathematical concepts in the primary school, and 'a fraction' is a mathematical concept, while a Cuisenaire rod is not. To be sure, Cuisenaire rods are effective tools in teaching a considerable amount of primary-school mathematics. They may well deserve attention in an in-service setting. But they are best dealt with in the context of how to teach this concept or that, rather than singled out for treatment in their own right.

In the light of the foregoing, it is recommended that:

Both the short-term and the long-term needs of teachers should be considered, but emphasis should be given to their long-term needs.

In-service courses should blend content and methods, bridging the gap between theory and practice.

In-service courses should be organized around mathematical topics and not around the teaching materials.

Classroom observations, preliminary meetings, pre-course questionnaires, post-course evaluations and pupils' performance data should be used to assess the in-service needs of teachers.

How can school systems, universities, teacher-education institutions and governmental agencies help to meet the in-service needs of teachers?

The problem of finding money to support in-service education is one that can be solved only by the authorities at the local, regional and national levels. It is especially hard if a school system is having difficulty in simply finding teachers to staff classrooms. There are no guidelines that offer practical guidance on how much of the educational budget should be invested in in-service education. Evidence in developing countries (Otte, 1979, Osborne, 1977; Skilbeck et al. 1977; and the National Council of Teachers of Mathematics, 1980) suggests that the provision of in-service education by the school system significantly improves the attitudes of teachers. If so, it is reasonable to infer that in-service education will serve to extend the tenure of teachers and so protect the initial investment of the education system in pre-service education. Furthermore, it should be noted that extending the length of service of teachers can greatly benefit the younger teachers if the older teachers give them a lead. In-service education gives training in leadership if it is well designed. So in-service education is cost effective on both counts: in extending the length of the tenure of teachers and in helping not only the older teachers, but also the younger ones, to become more effective.

In the earlier discussion of needs, it was suggested that in-service programme should be developed jointly by primary-school teachers and by those better able to assess the broad issues. School systems, institutions of higher education and government agencies can each contribute to the development of in-service education.

School systems can ensure the release of classroom teachers from their duties, thereby enabling them to plan and to participate in in-service programmes or professional meetings. They can also provide facilities and materials for on-site courses, and they can maintain professional libraries that will include appropriate journals. They can help to publicize in-service activities, and encourage participation in them. In short, school systems can provide the financial support that is necessary if teachers are to participate in in-service education.

Teacher-training institutions should recognize that it is their responsibility, and an important one, to contribute to the development of in-service education. In most countries, the pursuit of academic work beyond that required for initial certification is a primary means of advancement to greater responsibilities within the school system, to higher levels of certification and to increased pay. Thus, continuation in higher education is for most teachers an important mode of in-service education.

However, many institutions of higher education have difficulty in designing courses that cater for the in-service needs of teachers. 'Academic respectability', including, as it does, research, institutional tradition and the so-called 'academic standards', may seem to be in conflict with courses for teachers. This tends to inhibit the participation of university staff members in in-service education, particularly in the content areas. Typically the commitment of the majority of staff of an institution of higher education is to teaching and to research. To provide teacher education beyond that needed for initial certification is but a small part of the total work of the institution. Faculty members in academic departments may have little inclination to help teachers who have been away from academic work for some years and who may need to renew their perceptions of mathematics, psychology or theories of instruction. For example, to provide a course that will be relevant and meaningful to teachers whose mathematics is 'old and cold' can seem incongruous to a faculty member who has been working with students in higher mathematics. Yet the university cannot shirk its responsibility to schoolteachers. Indeed, it is in the university's self-interest to do as much as it can to ensure that its students will have the best possible background when they enter.

Institutions of higher education should provide courses that satisfy both the need for academic work at an elementary level and the need for university course credit. This may be done by examining relevant elementary ideas while maintaining intellectual integrity. For example, it is perfectly possible to devise an approach to the study of decimal fractions that is quite informal, yet highly challenging.

The discrepancy between teachers' needs and the provisions of the traditional course at institutions of higher education may be further diminished by distinguishing between the standards required for certain courses and those for an advanced degree in mathematics. The distinction would allow the primary-school teacher to earn credit in mathematics that could be applied to certain degrees of the university other than to a degree in mathematics itself. School systems can help by recognizing appropriate coursework in mathematics as legitimate professional improvement.

Institutions of higher education should examine the accessibility of their in-service programmes to teachers. Registration procedures, assignments of staff, the availability of facilities and the courses offered can be designed to encourage teacher participation.

Government agencies have an important role in in-service education. They can provide direct help or help in planning to local authorities by supplying expertise and resources that are not available locally. For example, the Ohio Department of Education recently developed two booklets to help teachers to teach problem-solving (Meiring, 1980). This flowed from the recommendation made by the National Council of Teachers of Mathematics (1980) that problem-solving deserved significantly greater emphasis in the curriculum. The booklets are designed for self-study, but they would be quite useful in an in-service course setting. They provide numerous classroom-based examples of heuristic strategies for problem-solving, and they introduce teachers to the theoretical and research bases for designing instruction on problem-solving. This resource for teachers was developed in response to a particular need spanning many local school systems. It would have been difficult to meet through courses provided at the local level.

The continuing education of teachers is important for a variety of reasons. Schools should reflect a national purpose in meeting needs and providing for the development of a nation and its citizenry. However, regional and local variations in need are important and have to be catered for. Schools in the coastal regions of Nigeria are preparing students for a different way of life, with different mathematical needs, from those with which schools in the mining or agricultural areas of the hinterland are concerned. Instructional projects should reflect these regional variations, and the corresponding in-service education for teachers should be developed accordingly. Resources for in-service education should be allocated so as to allow attention to both regional and national needs. Further, within many countries there are often profound differences in the capabilities and commitment of local education authorities to provide support services for teachers. We suggest that the governmental assistance for in-service education should reflect these regional differences.

The provision of in-service education for teachers depends on policy, and upon there being resources for the support and maintenance of programmes. For these, education must compete with other societal demands. Most countries and states have limited resources to devote to health care, industrial development, agriculture, defence and other services. The individuals who determine the priorities in the use of resources need information if they are to make rational decisions. An important responsibility of education agencies at all levels is to provide decision-makers with the information they need to justify their committing support to in-service education.

Developed countries, with a long tradition of providing education for all children, have evolved the practice of placing responsibility for much in-service education upon the personnel of the school. This is seen as a part of the professionalism of leadership within the schools. Although such school systems also look outside for sources of ideas, a major responsibility for in-service is accepted as the school system's own. The evolution of school systems towards generating indigenous leadership and in-service education appears to be a natural and sensible way forward. The long-term planning for in-service education should encourage this natural trend.

The foregoing discussion suggests the following recommendations:

School systems should release teachers from their duties so as to permit them to plan and to participate in in-service work and to attend professional meetings.

School systems should provide facilities, professional libraries and materials for in-service programmes and should assist in publicizing in-service opportunities.

School policy on promotion and compensation should recognize appropriate in-service experience, be it in mathematics, in related disciplines or in mathematics education, so long as the experience contributes to the professional growth of the teacher.

Universities and teacher-education institutions should recognize the development and implementation of in-service education as an important part of their responsibility.

Universities should provide courses in mathematics and related areas that are stimulating and challenging while being relevant and accessible to primary-school teachers, who may have weak backgrounds in mathematics.

Universities and teacher-education institutions should make in-service programmes readily accessible to teachers through the careful choice of staff, time, location and ease of registration.

Governmental agencies should assist in the support and development of in-service education, including the production of useful instructional materials for the in-service course.

In-service education should reflect regional as well as national needs.

Education agencies within the government should provide legislative and administrative bodies with the information they need to develop policy and allocate resources to support in-service education.

School authorities should aim to promote qualities of leadership for in-service work within the school system so as to protect their investment in pre-service education.

How should the problems caused by cultural and economic differences that arise when establishing in-service programmes for developing countries be dealt with?

The president of the Inter-America Committee on Mathematical Education, Ubiratan D'Ambrosio of Brazil (1981), contrasted education in developed countries with that in developing countries as follows:

> In most developed countries school systems are already fully established, attending to the needs of a somewhat stable population. In these countries evolution of the educational system means basically improving what already exists. In a sense, the objective is to run an ongoing operation, albeit in a better way. Hence change cannot be profound. The existing structure–mainly the teacher power structure in the classroom–is established and very difficult to change.
>
> In developing countries, the educational system is a system in the making. The risk of building up a system which may be obsolete at its birth invites a deeper analysis of what to expect from an educational system as a whole. The problems of a global nature which permeate the process of development require comprehensive planning for education involving all of a nation's social structures: its people, its societies and its cultures. This planning must establish future priorities in terms of current needs.

Bienvenado Nebres (1981) of Ateneo de Manila University indicates a South-East Asian perspective in observing the reliance on developed countries for mathematics education:

> The paradoxical aspect of these differences (especially the scarcity of human and material resources) is that instead of isolating us from developments in advanced countries, they make us more vulnerable to them. This is because we have to depend on Western mathematics educators and Western textbooks. We do not have the necessary number of experts nor the funds to develop our own textbooks. So our textbooks are either completely Western or are minor adaptations of Western textbooks. Similarly, we depend on outside experts for our in-service programmes for teachers, for advice on trends and recent developments.

D'Ambrosia and Nebres ennunciate a critical issue that is reaffirmed by the authors' own experiences: given the need for a deep understanding of a developing nation's problems, to what extent should the 'experts' from developed countries assume the responsibility for making their ideas conform to the conditions of the developing country?

To illustrate: to what extent should a mathematics educator from the Netherlands, when working in Indonesia, adjust her own position on a child-centred approach to learning by making her presentations in Indonesia conform to the there more prevalent teacher-centred approach? Should a mathematics education consultant from the United States, when doing in-service work in Costa Rican schools, strongly encourage the use of calculators when calculators are not currently available? Even though the consultants may be experts in mathematics education, do they understand the cultural and economic nuances well enough to make appropriate adjustments? Should they present what they think is best and assume that the leaders in the receiving country should adapt what is advocated to the cultural influences and economic constraints?

There is a parallel between the situation that can occur when the recipients of an in-service course in a developing country are more sensitive to local needs than are the consultants from the developed country and the situation, discussed earlier, in which classroom teachers are more sensitive to local needs than are the faculties of universities or teacher-training institutions. In both cases, the barriers to understanding can be reduced by improving the communication between the trainers and the trainees.

Communication with visiting consultants may be improved by:

Advance briefing of the consultants about cultural and economic factors and the goals of education before they depart from their home country.

Observation in the local schools by the consultants soon after arrival in the host country.

The interaction of visiting consultants and leaders from the host country in the development phases of the in-service programme.

Assigning the consultants to teams to accelerate their learning experiences.

Retaining the consultants as resource personnel after their return home.

A vital ingredient in consultancy work in a foreign country is the ability of the consultants to describe alternatives. Both the strengths and the weaknesses of new approaches should be presented. For example, an authority on Competency Based Teacher Education (CBTE), when working in a country where the approach is new and unfamiliar, should elaborate on the arguments against CBTE as well as those for it. Alternatives to CBTE should be described. The consultant should recognize that the professionals in the host country have to make intelligent decisions between alternatives, since they will have to carry the responsibility long after the consultant has returned home.

Another element that is common to both international and local in-service work is the need to focus on long-term needs. While visiting consultants play an important role in assisting a developing country, the long-term goals of the country can be better achieved by the thorough preparation of indigenous leaders. This preparation may necessitate study abroad. A core of well-prepared mathematics educators within a country can provide in-service education that is sensitive to the country's culture and economy. The developing country must recognize that using foreign consultants and sending their own professionals to other countries to study drains resources. Eventually a country should depend upon its own professionals to provide the leadership for in-service education.

The issue of deciding the extent to which the curricula of developed countries should be responsive to needs of students from developing countries arises when considering studies abroad. In discussing the relevance of American graduate curricula for foreign students, the Council of Graduate Schools in the United States (LaPidus, 1980) holds that 'the general question of whether programs should be modified to accommodate the interests of foreign students can be answered only in specific terms, and only in the context of certain basic characteristics of American graduate education.' The committee cites the discipline structure, the type of degree and the faculty advisor as factors in determining the programme. Furthermore, 'the extent to which a student's graduate experience is consonant with that student's own goals is dependent on his ability to articulate these goals, the flexibility available in degree requirements, and the faculty advisor's willingness to accommodate the specific interests of the student'.

Whether the in-service courses are given in the developing country or in the developed country, the cultural and economic differences between the countries must be taken into consideration. The problems and issues inherent in these differences suggest the following recommendations:

It is a responsibility of international consultants to acquire information: About the site of their in-service work, and about the culture, the educational traditions and practices, the economics and the goals of the school system of the host country before they arrive. Conversely, the school system of the host country has a responsibility to provide this information to the consultant.

The host country and the international consultant should give high priority to acquainting the consultant with the school situation, including the teachers' capabilities, their needs, the instructional tools that are available and the educational goals of the system.

International consultants should make it a point to set out alternatives thoroughly enough to allow rational decisions between the alternatives to be made by the host personnel.

The responsibilities of outside consultants should not end with their leaving the host country; they should serve as resource personnel after returning to their home country.

The design of in-service programmes should recognize the long-term need of the school system to build the capability of providing local leadership in in-service education.

Programmes of intensive study at leading institutions should be provided to prepare leaders from developing countries.

Concluding remarks

In-service education contributes to change in primary-school mathematics (Price, 1981) and to the professional growth of primary-school teachers (Otte, 1979). Two fundamental premises have guided the choice of recommendations: (a) the in-service programme should be closely related to the mathematics to be taught; (b) the in-service programme should seek to augment local leadership.

The discussion in the preceding section of the processes and responsibilities of international consultants provided a setting for these two premises. The extreme case of the international consultant is, in fact, remarkable in that the processes and principles that apply to him apply equally to any outsider involved in an in-service course. The professor from a local university and the inspector for mathematics in a province or a state must be conversant with the local school situation if they are to provide effective in-service education. The ability to listen and to think are the most important requirements of any in-service educator. School systems must go elsewhere for ideas to stimulate creative local leadership.

In-service education costs money. However, it is a sensible investment. Teaching can become demoralizing if teachers encounter the same problems year after year but make no progress in resolving them. In-service education can help teachers to overcome their teaching problems. So in-service education helps to give assurance to teachers, who communicate the joy of learning and who retain their initial commitment to teaching. Investment in in-service education is a wise use of resources in that it helps the classroom to operate more efficiently.

In-service education must now be considered a continuing feature of the professional life of a teacher. It is not possible in a pre-service course to learn all that is useful in mathematics education. This would be so even if the course were devoted exclusively to mathematics education for six years. Because of the growth of knowledge in mathematics education and the new tools for doing mathematics, the calculator and computer, the responsibility of the teacher for continual learning will increase in the future. Pre-service education, and the orientation of a school system to continued learning, should lead teachers to expect in-service education to become a frequent, normal part of their professional life. Teachers should be paid a wage that allows them to invest in their own professional growth. Teachers should also expect their education system to provide support for and to give opportunities for in-service education.

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Bryn Roberts

The in-service teacher-education project in Swaziland (1973-77)

This account of the primary in-service teacher-training project in Swaziland is based on a personal experience as a mathematics tutor on the staff of the project from 1973 to 1979. This period marked the first phase of the project, when 600 unqualified and underqualified teachers were certificated as primary-school teachers.

The account is written in three parts: the structure of the in-service course, the mathematics aspect of the in-service course, and the role and function of in-service education in developing countries.

The structure of the in-service course

By the beginning of the 1970s, owing to a gradual increase in the number of primary schools, the number of untrained teachers working in Swaziland's education system had reached an unacceptable level. These teachers were employed, in the main, to teach the first two or three years of primary school. This measure, in itself, often resulted in those teachers who had been trained specifically for those early years being placed with older children. To counter these trends, a project, jointly agreed by Unesco and the Government of Swaziland, was drawn up in 1971. Its objectives were:

> to improve the teaching strength of the country by training some 600 primary-school teachers; to inculcate modern methods, not only for use by the trainees, but also as a medium for change in others; and to organize the training in such a way that it caused a minimum of disruption to the staffing of schools at any time. (Nsibandze and Green, 1978, p. 110.)

A project with similar aims had been attempted five years earlier in Botswana. From it, lessons had been learnt about the development of project staff and about school-based in-service strategies. These vital elements were built into the Swaziland project, and it was decided that: the staff of the project would be appointed six months in advance of the beginning of the actual teaching programme, so that they could collect information and develop plans; and follow-up visits to the trainees' schools would have to be carefully organized and tenaciously maintained.

The style of the project, like that of its predecessor in Botswana, was innovative in nature. It comprised a mixture of in-college and school-based methods. These were: a three-year course, with three in-college courses each of six weeks duration; three sets of eight assignments (to be mediated by correspondence) in English language, education, mathematics, science and social studies; tutorial supervision and assistance for the trainees at their schools; and a programme of radio broadcasts.

By January 1973, all the staff needed for the project had been appointed, and were based at the William Pitcher College, Manzini, where the project was housed. They were sixteen in number, and they comprised the project manager, with overall responsibility for policy and management, a head of correspondence studies, with specific responsibility for the production and marking of correspondence assignments, two tutors each for language, mathematics, science and social studies, two typists, one collator and two drivers.

The staff travelled extensively throughout the country, informing teachers and headteachers of the innovative programme that was soon to be put into effect. In so doing, they became familiar with the nature of teachers, the children they taught, and the society they themselves were becoming part of. It was found that untrained and underqualified teachers were very keen to gain more knowledge and become more able teachers; there seemed to be little staff interaction, and little contact between headteachers and their district education officers; it also seemed that very little happened, educationally (formally or informally), during the first four years of primary schooling. In general, the feeling amongst teachers and headmasters was that things would begin in earnest in Standard 3 (Grade 5), when they would begin to prepare the children for the Standard 5 (Grade 7) 'Primary Leaving' Certificate. The impression, too, was that school resources of furniture, teaching equipment and books were inadequate. Moreover, class size varied between five and fifty, according to location and type of school. The norm was thirty-five to forty per class.

The staff made valuable contact also with school inspectors, education officers, and the country's newly formed Primary Curriculum Unit (PCU). Particular note was taken of the work of the latter, for it was in the process of collecting the views of people at different levels in Swazi society on what they thought the aims and objectives of primary education should be.

Following these visits to schools, and after gaining information from others in the education system, discussions among members of the project's staff led to the following strategies being agreed:

The in-service course would be aimed specifically at teachers in the first four years of primary school, and headteachers would be encouraged to use the knowledge of these teachers throughout the course after they had become qualified.

In order to give teachers an insight into effective interaction with small children, the correspondence courses written by the staff would emphasize the use of group work and the discovery approach. During in-college courses, teachers would, as far as possible, be taught by methods that they themselves could use in the primary classroom. The workshop approach, using worksheets for mathematics, science, social studies and, to a lesser extent, for English, would be vigourously applied. Discussions on the various styles and approaches to teaching, including child-centred education, would take place in 'education' tutorials.

A demonstration/practising school would be set up by the staff. The visits of in-service trainees would include observation of group work, and of practice in the use of group work.

Due to transport constraints (two vehicles to cover the whole country), the staff would visit teachers in their own schools to assist them and assess their performance three times a year. Strenuous attempts, however, would be made to see the in-service trainees as soon as possible after their course, primarily to give assistance with classroom organization and management. Newsletters and twice-weekly radio broadcasts would provide additional contact.

It was decided that the project's staff would seek to be involved with the various components of Swaziland's PCU, and serve on committees if and when requested to do so.

By forming and enunciating these strategies, members of the staff were moulded into vigorous 'change agents', whose commitment to the project increased. Moreover, their confidence was enhanced as they became collectively responsible for the project's objectives.

The mathematical aspect of the in-service course

Various strategies for the mathematics course emerged from the research undertaken by the two mathematics tutors. These were:

Most of the in-college course sessions would be of the workshop type involving group work with equipment or apparatus.

The correspondence work would initially provide ideas on classroom management and organization. It would then proceed to cover areas that were new to the trainees, such as number bases, together with topics the participants requested and those they seemed to lack knowledge of.

The course in mathematics for the first four years of the primary school would be written by the staff, and be based partly on the objectives formulated by the PCU. These courses would be based on a discovery approach, utilizing group activity with a defined set of materials, equipment or apparatus. Each trainee would return to his or her school with that part of the course applicable to his or her particular class.

The trainees would return to their schools with simple apparatus they had made in college.

Extra courses in local areas would be conducted. Headmasters, as well as the in-service trainees would be invited to attend. This would allow the headteacher to see what was expected of their teachers, and help to identify areas where they would be able to provide professional assistance.

An equipment and materials centre for mathematics (and science) would be set up in the in-service department of William Pitcher College with Ministry of Education financing. Equipment and materials would be sold at a subsidized rate, and headmasters would be encouraged to use their school funds to purchase those items which the trainees requested for their classrooms.

Visits to schools would be viewed as opportunities to give assistance with classroom organization and planning, to encourage trainees to evaluate the effect of some work carried out before the visit, and plan future arrangements for mathematics activities.

Radio broadcasts would be used to discuss in general terms what the mathematics staff had observed while visiting schools.

These strategies for the in-service course in mathematics existed at two levels: the level of the in-service course itself, and the classroom level at which the trainees would operate. Two broad issues arose in the course of putting them into effect, that of content and communication and that of evaluation. These two issues will now be discussed.

The mathematics content of the in-service course was based on the country's primary-school mathematics syllabus. But note was taken of the emphasis placed on certain areas of content that had a bearing upon PCU's objectives. The areas of content were seven in number, namely, shape, number (sets), length, mass, time, water (capacity) and shop (money). The equipment needed for these topics was set out on benches around three sides of the mathematics room of the college and appropriately labelled. In the open area of the mathematics room, there were five work tables. During a working session, the equipment would be taken from the side benches to the work tables, where five groups of trainees would work. Their tasks were directed by 'worksheets' and a 'rotation' scheme. Each rotation of five worksheets was called a cycle, or unit. The purpose of this arrangement was to simulate a school situation of five groups of children working for a 'week' of five school days.

Each worksheet provided information and direction regarding equipment and materials (paper, crayons, etc.) to be used in the initial inquiry stage. Thereafter, there would be questions to lead the trainees on to a further aspect of a topic and eventually on to 'practice'. In general, a series of worksheets based on the same topic area would sequentially take the following steps:

1. The use of 'concrete' materials.
2. The use of a 'model' (an arrangement of 'concrete' materials).
3. The use of a picture or diagram (the representation of step 2).
4. The use of symbols (e.g. numerals).
5. Practice in steps 2, 3, and 4.
6. Practice in step 4.

The need for step 1 with teachers deserves a short explanation. It derived from the fact that the algorithms, which may emerge in steps 4 and 5, were usually familiar to them, but not fully understood. So it was necessary to drive home to the teachers the idea that the sequence of modelling (step 2), followed by representing (step 3), followed by symbolizing (step 4) is a powerful and acceptable tool for children to use in coming to terms with mathematical ideas. By gradually coming to realize the potency of this sequence through the experience of following it for the algorithms they themselves learned in primary school, the teachers were usually prepared to use this mode in their schools, if given the opportunity.

The equipment needed in the college mathematics room to promote this development and its transfer to primary schools comprised many varied items. Most of them are recorded here, topic by topic:

Shape: jig-saws, insert boards, solids and plane templates, peg boards, geoboards, plasticine or clay, scrap materials, string, scissors, glue, tangram sets, tessellation sets.

Number (and sets): counters (bottle tops, seeds, beads, matchsticks, etc.), Cuisenaire rods, centicube, base-10 blocks (cube, long, flat, block), attribute blocks, abacus, maths balance, number games (dominoes etc.).

Length: metre rulers and sticks, height measurer, trundle wheel, tape measures, string, area grid in cm^2.

Mass: arbitrary and class masses, metric masses, equal-arm balance, automatic balance, bathroom scales, centicube.

Time: clock, clock-faces, stop watch, 10-second timer, pendulums, roller track, sundial, sand-clock.

Water: arbitrary and class capacity measures, metric measures, spoons, cups, bottles, plastic containers, bowls, basins, buckets.

Shop: shelving placed behind a counter on which a till is placed, empty packets placed on the shelves to 'buy', price tags, balance, plastic, card or real money.

Each in-service group returned to the schools with a list of materials needed for their classes, together with a scheme of work in mathematics for those classes. Many teachers used their own money to buy the equipment they required rather than wait until their headteachers decided to use school funds for that purpose.

Evaluation and assessment

Specific instructional objectives for the worksheets were not ennunciated. The emphasis was on doing things rather than on the reasons for doing them. Such reasons could be discussed informally while the worksheets were being followed by groups of teachers.

The mathematics tutor's task, during these sessions, was to sit with each group of students for a reasonable length of time to assist or to realign investigations and to assess the attitude of the teachers to the six-stage development. There would often be the need, in response to a question from the trainees, to discuss the value of what was being done. During such times, it was necessary for the tutor to listen carefully, especially when a trainee described his or her teaching situation and how the new approach could be used in that situation.

During the in-college course, emphasis was on the formative assessment of the trainees, and particularly their attitude towards promoting changes in their classrooms. In the rather open atmosphere of the mathematics of the college, a professional relationship could grow, instead of the usual lecturer/student relationship.

The trainees were summatively assessed on their responses to the correspondence course. This involved sending assignments out to them in their schools and their sending back to the college answers to questions set out in the assignment. The first set of eight assignments in the correspondence course was directed towards assisting the in-service teachers to produce an environment in the schoolroom that would, for young children, be conducive to learning. There was an emphasis on children using the equipment, and contributing to the overall classroom atmosphere by producing pictures, etc. for the classroom walls. The first eight assignments covered shapes, number and sets, money, length, mass, time and the mathematics classroom. Each assignment provided background information on teaching the topic and advice on making and buying equipment. The assignment on the 'mathematics classroom' discussed its general organization and management. Both sets of the eight correspondence assignments given in the two subsequent years were addressed either to those topics in which the trainees requested assistance or to those in which they were considered to be weak.

As has been mentioned earlier, strenuous attempts were made to visit trainees in their schools as soon as possible after their first in-college course. This strategy gave the teacher the opportunity to introduce the headteacher to the tutor, who could then explain on the spot the aims of the in-service course and what was expected of the in-service teacher in the course of his or her training. During these initial visits, the teacher was assisted in setting up the classroom and in arranging the furniture for group use. Subsequent visits during the trainees' first year course developed this, and the trainee was assessed on the quality of the physical learning environment that had been produced in the classroom. Visits to the trainees during their second and third years tended to dwell more on what was actually being taught and how.

In this school-based in-service training, worthwhile interaction between tutor and trainee could be based on seemingly simple, but very important questions, such as:
Which worksheets are you using this week?
How do the children react to the worksheets?
What do you think the children are learning?
How could the worksheets be improved?
Which workcards will you use, next week?
These questions provided the tutor and the trainees with a framework of thought within which a variety of things could be discussed. For example: mathematical structures, sequence, progression, remedial and enrichment exercises; classroom organization and management; the use of resources and sharing resources; resource requirements; and relationships and discussions with other members of staff.

The role of function of in-service education in developing countries

Effective in-service education should have a dual role—improvement of the individual teacher and thereby improvement in the education system. To this end it is necessary that the individual is fulfilled by the experience and that the system can accommodate the new skills that have been learnt.

Michael Eraut (1972) points to three important functions of in-service education: to provide new knowledge, to facilitate professional discussion and to produce an innovation in response to an education problem.

The third of these, in the case of the Swaziland project, occurred in four parts or stages: looking at schools, identifying an education problem or problems, creatively designing a course to meet the problem and incorporate Eraut's first and second function, and action to enable the education system to use an innovative approach incorporating new skills learned by teachers.

The thrust of in-service education must be felt in the schools. In-service education, therefore, should have a strong school-based element. To create this school-based element, it is essential for the in-service staff to meet with teachers in their schools and discuss education problems with them on a professional level. A mixture of in-school activity and in-college sessions is highly desirable in the process of increasing professional awareness, increasing skills and promoting improvement within the system.

The Swaziland project improved the skills of many teachers and it made a positive contribution to improved schooling for the children. It may be said that the mathematical aspect of the in-service course played its part in motivating teachers, although perhaps no more so than the courses given in other subjects. In some ways, however, the mathematics course was an ideal vehicle for promoting change, for it provided opportunities for teachers to contribute to classroom management and organization, to learn to use resources and to improve teaching in schools, to co-operate with other members of staff in implementing the primary-school syllabus, and to become more skilled and so to contribute to the overall improvement of the education system of Swaziland.

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Maria do Carmo vila and Reginaldo Maves de Souza Lima

A case-study of an in-service training, programme for practising teachers without degrees

Distance training for teachers

The training programme described here has been developed in the inland Brazilian State of Minas Gerais, where approximately fifteen million inhabitants live. The capital, Belo Horizonte, has a population of about three million. The programme is expected to reach over ten thousand teachers.

The Federal University of Minas Gerais (UFMG) is located in Belo Horizonte. It is funded by the federal government and is the largest institution of higher education in the state. It houses two centres that have a direct relationship with the programme. The first is the Pedagogical Centre, a primary and secondary school run by UFMG for educational purposes and used as a pedagogical laboratory. The second is the Science Teacher Training Centre of Minas Gerais (CECIMIG). This is located on the UFMG campus, but it is not funded by the university. Many of its specialists, however, are from UFMG. CECIMIG has been training science and mathematics teachers for the entire state since 1965 and is responsible for running the-teacher-training programme under consideration. Instructional materials developed for the programme are first used in the Pedagogical Centre on an experimental basis. Then, if approved, they are used in the training programme.

There are three levels of education in the Brazilian system, namely the elementary and lower secondary (eight years), the secondary (three years) and the tertiary. A secondary-school education is a prerequisite to teach children in the first four years of the first level. Otherwise, a tertiary level education (or higher) is required.

Problems encountered

The main problem confronting any attempt to retain practising teachers is one of numbers. In Minas Gerais alone there are about 30,000 teachers involved in mathematics. But there are no more than thirty departments of mathematics in tertiary level institutions. To retrain these teachers by traditional means is logistically impossible. It would, moreover, be a form of economic suicide to try to do so: there is not enough space, not enough money and not enough competent staff.

Instructional material

In 1977, work began on the programme and it was organized in a sequence of eleven stages. The sixth stage is now being initiated. The main concern from the start has been that of obtaining the financial resources needed to carry out the programme.

As to instructional material, policy has been to keep to a minimum the quantity of material for students of the teachers enrolled in the programme. Stress has been on materials for the use of the teachers themselves. These emphasize both content and the methods of teaching it. Both the materials and the organization of the retraining programme are firmly based on educational concepts that seek to lead to a new type of teacher/student relationship. Through this, it is hoped that the student will gain a maximum advantage from mathematics. The guiding principles that provide a basis for the preparation of the material are as follows:

Think of the mathematics lesson not only as a mere transmission of content, but also, and most importantly, as an opportunity to teach through the use of mathematics.

Avoid the growing tendency to transform mathematics teaching into preparatory sessions for specific examinations or selective tests.

Introduce new themes through the students' environment, or through the simulation thereof, but never through content already seen; to do so verbalizes the teaching and makes learning more difficult. In short, assume that too much initial mathematics kills mathematics.

Approach a new topic from the problem point of view, not from the information point of view.

Emphasize problem-solving at each level of activity in order to challenge the student's imagination continually.

Give the student freedom, from the start, to create his own strategies for problem-solving, and, most importantly, provide incentives for this practice instead of furnishing ready-made methods, since life requires that the individual make decisions, solve problems and transfer knowledge.

Lead the student, who has thus created his own strategies for problem-solving, to refute his own ideas and his own solutions, for, if life depends on the courage to act, our growth is as great as our ability to criticize.

Teach the student to search for information on the subject being studied from sources other than those used in the classroom.

Help the student to organize the information he obtains–from the classroom and from other sources–so that he may then, and only then, memorize it.

In order to translate these ideas into tangible form, it was decided to develop two broad types of material: material prepared exclusively for the teacher and material prepared for the use of the child, but with which, of course, the teacher must also be familiar. For the children, cards, cut-out material, worksheets and 'plaques' were prepared. For the teachers, the materials are of three types: didactic, enrichment and psycho-pedagogic. The didactic material is in the form of a comic strip. This form has proved helpful for elementary-school teachers who have trouble in following directives in straight text. It is also a technique that facilitates the introduction to illustrated models and opportune dialogue. Comic strips can greatly help the trainee. By simplifying the information, the didactic material presents the lessons that the teacher will give, bearing two needs in mind, namely:

The need to guide the child, so that he himself will create his own strategies (although we do not lose sight of the Piagetian contention that the strategy of the child is extremely simplified).

The need to introduce each and every subject at the elementary-school level as a real subject; this is based on the fact that the child will not be at the stage of being able to perform abstract operations and will not therefore, be able to reason with hypotheses.

In this situation, multiplication, for instance, is not taught on the basis of addition, nor is potentiation taught on the basis of multiplication. This obliges the teacher to introduce each subject with pre-mathematical activities, through which the student solves problems within his reach, and then goes on to extract important information. The teacher is a guide, an orientator.

When ready for trial, the material was used in the Pedagogical Centre. It was then revised in the light of student reaction and the teachers' recommendations. The material was subsequently tried out by groups of teachers from various regions in 'in-presence' courses. These courses employed a technique that was called 'teaching by correspondence inside the classroom'. Each day, four trainees would be given didactic material and student material. They would study them and use them the next day on their colleagues, without outside intervention. The organizers observed their work. Afterwards, everyone would comment on the work and on the material.

This simulation of a 'distance' course led to a further improvement of the material, a more economical method than evaluating the material after an actual course.

As has already been said, the programme unravels in a sequence of eleven stages. The objectives of these stages are as follows:

1. To develop instructional material designed to train teachers, covering eventually the teaching of the first four grades.

2. To train a small number of teachers so that they may implement, on an experimental basis, in some community schools, the material developed in Stage 1. In this, some state and private schools of Minas Gerais participated.
3. To revise the instructional material tested in Stage 2, putting it into its final form for the long-distance training of teachers, and for possible implementation in the first four grades. In this also, state and private schools of Minas Gerais participated.
4. To acquire graphic equipment for printing materials developed, tested and revised in the previous stages.
5. To train 'monitors' (staff from superintendencies of education of the State of Minas Gerais) through 'in-presence' courses in Belo Horizonte (where first- to fourth-grade instructional materials were utilized) and to follow up these courses with distance training of the same personnel.
6. To train 10,000 mathematics teachers of grades 1 to 4 within the structure of the programme. The steps foreseen at this stage are the printing of material, the publicizing of the training programme, the training properly so-called and direct CECIMIG orientation of mathematics teachers by means of magazines, information bulletins or new materials. Here, in parenthesis, it should be said that it is this stage of the project for which funds are currently being solicited. It is encouraging that some faculties and universities located in the interior of Minas Gerais (and in other states as well) have already taken action to obtain funds in order to become associated with this training programme.
7. To develop instructional materials for grades 5 to 8. In this connection, it can be said that many materials suitable for the teachers of these grades, as well as some materials for the students of these teachers, have already been used in the Pedagogical Centre. The task now is to realign the development of materials within the perspectives of the programme.
8. To train 'monitors', as in Stage 5, in the use of the materials developed in Stage 7. At the same time, continuing training will be given to teachers of grades 1 to 4. In addition, the development, implementation and reformulation of secondary school materials will commence.
9. To train, at a distance, teachers of grades 5 to 8, within the framework of the goals of the programme. Steps similar to those envisaged in Stage 6 are foreseen here.
10. To train 'monitors' in the use of instructional material for the secondary-school level through 'in-presence' courses and distance training as before. Simultaneously, continuity will be given to the training of teachers of grades 1 to 8.
11. To train mathematics teachers at the secondary level, within the

framework and objectives of this programme. This, too, will involve printing material, publicizing the training programme, training properly so-called and direct CECIMIG orientation of mathematics teachers by means of magazines, information bulletins or new materials.

The programme also foresees work on educational technology. This will begin with the preparation of video cassettes depicting the activities of teachers and students of the Pedagogical Centre when using the methods adopted by the programme. This material would be placed in the superintendencies or the faculties taking part in the programme. It would serve as a source of stimulus and information for the trainees. The production of audio-cassettes to orientate the trainees is also envisaged. These would encapsulate lectures on teaching, studying, and on student/teacher interaction.

In Formiga, a city 180 km away from Belo Horizonte, support will come from the city's Faculty of Sciences and Letters. The faculty has already named teachers to serve as 'monitors'. In Belo Horizonte, the City Hall and the State Secretariat of Education will provide 'monitors' to assist in the work. In ten other superintendencies, thirty 'monitors' have already been trained who will provide assistance. CECIMIG will maintain a schedule of teachers who will be responsible for the largest part of the work.

In conclusion

It is important to mention that trainees have to pay a fee to cover the minimum expenses of the training and the materials they need. However, the choice of the units used remains entirely up to the trainee. He or she studies and applies the unit of his or her choice. But during the first two years, he or she must study and apply only one unit per semester, to avoid being overloaded with change. Any topics not studied will continue to be taught using a traditional teaching method.

Finally, evaluation does not involve giving tests or setting examinations. Trained in a new topic, the teacher will apply it and send back a complete report of his or her activities and of the results obtained, as well as the opinions of the students and of the director of his or her school. On the basis of this report, the teacher receives a certificate of competence to teach in accordance with the principles of the programme.

Benjamin Eshun, Klaus Galda and Peter Sanders

Support for mathematics teachers: teacher associations and radio

Introduction

In most education systems, primary-school teachers rank lower, both in status and in salary levels, than their secondary-school counter-parts. Yet in many ways their job is more difficult. They are not subject specialists. They are expected to provide adequate instruction for their pupils in a range of subjects right across the curriculum. Moreover, the subjects themselves do not remain the same, and primary-school teachers are subjected to a constant bombardment of curriculum revision and change. Their conditions of work are less favourable. They face the same group of pupils and have to look after them throughout the day, rarely enjoying those free periods that provide a modicum of relief for secondary-school teachers. Furthermore, they often work in cramped or outdated buildings with very limited supplies of books, equipment and teaching aids. Many teachers have to take care of large groups of students, often at different levels.

While the needs of primary-school teachers are greater than those of their secondary-school colleagues, it is, in general, more difficult to help them. The problem is simply one of numbers. Because of the structure of the population in most developing countries, there are more primary- than secondary-school teachers. Furthermore, since they are not subject specialists, any support provided for a given subject area will have to be directed at all of them. Thus the systematic retraining or upgrading is likely to be a major undertaking. It has, in a number of countries, become the function of an in-service college, or some similar institution. Mechanisms for in-service training on this scale will not be considered in this chapter. Instead, less formal alternatives will be examined, but the problem of numbers remains. To take one example: it is not difficult to run a workshop for twenty secondary-school mathematics teachers on making teaching aids. But what do you do if 150 primary-school teachers turn up, each of them wanting to make a basic kit of teaching aids?

In this chapter, then, we will consider in some detail, and through case-studies, two mechanisms whereby this problem of numbers can be addressed and primary-school teachers assisted, namely by subject associations and radio broadcasts. There are, of course, others, such as the use of television, or a network of teachers' centres, with staff to man them. But subject associations and radio broadcasts have the advantage of requiring little by way of equipment or manpower over what is already available. They are thus well suited to the conditions that obtain in many developing countries at the present time.

Associations of mathematics teachers

General

Associations of mathematics teachers can provide an important institutional framework for the professional growth of teachers at all levels and for the development of a classroom-based curriculum. By taking part in meetings, by leading small group discussions, by running workshops or by writing articles for newsletters, teachers can share experiences with their colleagues and learn the basic skills of curriculum development. These associations exist in most countries. Typically, they organize regular national and regional conferences and workshops, and publish newsletters and reports. They frequently arrange for the distribution or the sale of mathematical books and teaching aids at a reduced rate.

Some mathematics associations have had a major influence in the revision of mathematics curricula in their countries. On some occasions, they have initiated changes that have been subsequently approved nationally. On others, they have drawn up specific mathematics curricula for special needs and have provided their own examinations. However, there is a tendency for mathematics associations to concentrate on the needs of secondary-school teachers. This is often because most of the office bearers and active members belong to this category themselves. This neglect of primary-school teachers is highlighted in a report on the British associations, the Mathematics Association and the Association of Teachers of Mathematics, delivered at ICME IV:

> One problem which faces both Associations in the United Kingdom is how to develop their role vis-à-vis mathematics in primary schools. About 80 per cent of all teachers of mathematics in the United Kingdom work with children under the age of 11, but few of these are in any sense 'specialists' and almost all of them teach across the whole school curriculum. It is rare for a primary-school teacher to accept a commitment to one particular subject to the extent of wishing to join an Association such as the MA or ATM; and yet the work which they do is fundamental, and they are an essential part of the 'body mathematical'.
>
> Both Associations have sought to make provision for them through special publications, journal articles, occasional regional conferences and the like; but it is

participation, not provision, which is the justification for the existence of a Professional Association—and this is a problem which remains to be solved.

In other countries, the situation seems to be similar. For example, in Fiji, assistance to primary-school teachers by the Fiji Mathematics Association has been limited to occasional meetings, mostly in rural areas. Taking place usually in a district centre over a week-end, these meetings have included panel discussions on the syllabus, on teaching methods and on the use of certain teaching aids. Workshops to enable participants to make teaching aids have also been held. One or two group sessions at the annual national conference have also been directed towards the needs of primary-school teachers. But, in general, attempts to recruit as members significant numbers of primary-school teachers have not met with success, and the main thrust of the work of the association is still at the secondary-school level.

The Mathematical Association of Ghana

One professional association that has provided an intensive and systematic in-service education for primary teachers is the Mathematical Association of Ghana (MAG). Formed in 1962 by about a dozen university mathematicians and secondary-school mathematics teachers, MAG has maintained a rapid rate of growth. By 1977, its paid-up membership was over 550, of whom more than a third were primary-school teachers. The activities of the association through which primary-school teachers derive professional benefit are fourfold. They are the Annual National Conference, the regional conference, the Mathematics Teachers Certificate and the *Journal.*

The Annual National Conference

The Annual National Conference is one of the most important events on MAG's calendar. In 1977, for example, over 400 members attended. Increasingly, in the last few years, MAG has provided a programme that caters for the special needs and interests of primary-school teachers. Lectures and speakers for this group of members are invited from the Faculty of Education of the University of Cape Coast, from the Curriculum and Inspectorate Divisions of the Ghana Education Service (GES), from secondary schools and from teacher-training colleges. Since 1978, the conference has spanned four days during the long vacation in August. It provides lecture courses, talks, symposia, poster sessions and exhibitions.

A lecture course consists of a sequence of two or more lectures on a particular topic on either the content or the methodology of mathematics. These courses are intended to increase the primary-school teacher's knowledge and understanding of mathematical topics in, or relevant to, the syllabus, and to acquaint him with alternative teaching methods and strategies. In order to cater for the differing needs and interests of primary-school teachers, three or more different courses are scheduled together on the programme. This also enables the number of participants in each course to be kept within reasonable limits that will encourage meaningful discussions and interactions. But many members have expressed disappointment that they cannot benefit from all the courses.

The talks that are given during the Annual National Conference benefit the primary-school teacher in various ways. Some help him to appreciate, perhaps for the first time, the nature of the problems that face mathematics education both locally and internationally. Talks are also given on mathematical topics of general interest and on the applications of mathematics in such fields as science, engineering, industry, art and business. Some talks introduce games, puzzles and other activities, such as making interesting polyhedra that can be used in extra-mathematics activities in school. Other talks are on teaching and learning aids, especially those that can be easily made by teachers, and on the topics the aids can effectively illuminate. The talks also include some that are given by invited high-ranking GES officers.

The symposia give the primary-school teacher a chance to hear the views of colleagues in the higher institutions and in GES, and those of other professionals on current issues in mathematics education and in general. The lively discussions and conflicting opinions serve to convince the primary-school teacher that his own views on current issues are important and that he has a real contribution to make to the improvement of mathematics teaching.

Poster sessions provide an opportunity for anyone who has a topic or a problem of wider interest in any area of primary-school mathematics to lead a discussion group on it. Sessions have included discussions of changes in the order and scope of the prescribed curriculum. They also provide an avenue for feedback to those officers in the Curriculum and Inspectorate Divisions of GES who are responsible for primary-school mathematics.

Finally, publishers exhibit books and materials for teaching primary-school mathematics. This gives interested teachers the opportunity to order books for their personal use. There are also exhibitions of teaching aids, often of a very high quality, made by both teachers and pupils.

Other benefits of the National Conference are less tangible, but still very real. Primary-school teachers have an opportunity of chatting informally with their colleagues in other schools or institutions. Ideas are exchanged in a holiday atmosphere and most teachers leave with greater commitment to teaching mathematics and with a sense of the community of the profession.

Regional conferences

Branches exist in each of Ghana's nine regions; they organize one-day conferences, at least once a year, usually during term time.

The nature of activities at the regional conferences is similar to those of the National Conference. The lecturers and speakers are mainly invited from institutions within the region. There is more opportunity for the primary teacher to be involved in these conferences than in the National Conference, especially as a panel member for a symposium. Furthermore, attendance is usually higher than regional representation at the National Conference.

The regional conferences are also able to attract potential members through the award of puzzle-competition prizes for pupils. Several consolation prizes are awarded in addition to the top three, and a number of teachers attend the conference primarily to accompany their prize-winning pupils.

The primary-school teacher certificate

In 1973, MAG's Project Committee introduced proposals for organizing intensive and systematic vacation courses for primary-school teachers. The courses were to lead to the award of the mathematics teacher's certificate (first cycle) by final examination on the courses.

The courses were intended to give teachers an opportunity for self-advancement through a study of mathematics content and methodology. The content comprised sets, numbers, relations, functions, algebra, geometry, vectors, probability and statistics. The methodology included a study of primary curricula, the learning of mathematics concepts, methods of teaching particular topics, problem solving and evaluation.

A total of seven vacation courses were held between December 1973 and August 1978, each lasting at least two weeks. More than twenty teachers completed the course. Unfortunately, attempts by MAG to have the certificate awarded either by GES or by the University of Cape Coast proved unsuccessful. So a final examination was never set.

Subsequently, a revised policy on in-service education now provides more attractive alternatives for promotion than the mathematics teachers certificate. So the courses have, for the time being, been brought to a halt.

The Journal and Newsletter

The *Journal of the Mathematical Association of Ghana* provides another means for the primary-school teacher to improve his knowledge in mathematics and the teaching of it. However, the scarcity of articles in the journal on primary-school work limits its usefulness.

MAG's *Newsletter* was first produced in 1978 to provide a channel for informing members of activities at the national as well as regional levels. The *Newsletter* contains one short article of general interest and summaries of some of the talks and of the contributions to symposia and conferences.

Conclusion

We have seen that the structure of the teaching profession makes it difficult for subject associations to provide significant help for primary-school teachers. Indications of what can be done, by way of courses, conferences and publications, have come from a number of countries. They provide valuable guidelines for associations that may be contemplating similar steps in the future. The account of the well-intentioned but ultimately unsuccessful programme of formal courses given in Ghana may suggest that a more realistic offering to primary-school teachers would be a pattern of more informal meetings and workshops, given as and when resources and personnel allow.

Radio

General

This section deals with the role of radio in mathematics education and its relation to the classroom teacher. Although radio has had a long history of providing educational programmes, dating from the 1920s, the last decade has seen a massive extension of its use. Its full potential will probably only be exploited in developing countries; in wealthier countries, there are too many competing and easily available technologies, such as television and computers.

A number of governments as well as the major international development funding agencies have begun to consider more seriously the role of radio in helping to solve educational problems, both within the formal school system and in non-formal education. Radio can be used to help in solving two of the principal problems that almost all developing countries face—the lack of textbooks (as well as other educational materials) and the shortage of trained teachers.

The main advantages of radio are obvious and very important to developing countries. They include its low cost and universal availability. It is effective with illiterate or semi-literate audiences and in situations where oral traditions predominate. From the educator's standpoint, programmes can be easily reviewed, and they need not be dependent on the distribution of other materials.

The main disadvantages of radio in education are even more obvious. In traditional (Western) formal education, the support of visual stimuli has been virtually indispensable. However, research in the past few years has shown that this disadvantage can be overcome with a combination of good programme design and (optionally) the addition of a small amount of printed materials, or the use of the blackboard. Another disadvantage is that radio is essentially a one-way medium.

This problem cannot be eliminated completely, but it can be reduced by limiting the use of radio to those parts of the education process that do not require two-way communication, or by designing programmes in the form of a dialogue between the radio and the students. In this way, students may perceive the programme to be a two-way activity.

A classification of radio mathematics programmes

A radio programme to teach mathematics can take many different forms, and the support a teacher receives will vary accordingly. A minimal radio programme in mathematics is an enrichment programme. Typically, this type of programme would last 15 minutes and be broadcast once or twice a week. Its main purpose is usually to motivate students. It might cover historical or cultural aspects of mathematics, or perhaps introduce some mathematical games to the students. Usually these programmes can be used either in the classroom or at home. The advantage of radio for this type of programme is that it makes available a large number of resources—e.g. music and professionally-produced drama—to which a classroom teacher would not have access. While enrichment programmes do usually try to teach something as well, their main purpose is to present mathematics in an enjoyable manner, thereby overcoming to a certain extent traditional student dislike of and resistance to learning mathematics. A good enrichment programme can be a valuable asset to a classroom teacher, who is often unable to keep students motivated. This type of programme has been exploited, with varying degrees of success, in many different countries.

An intermediate type of programme consists of a short series of broadcasts, usually designed to teach a particular topic. This is most often a topic that may be new, or exceptionally difficult for teachers. This type of programme may serve as a short training course for the teacher while the students are learning the material and at the same time provide a model of how the topic might be taught. These programmes can be used when a few new topics are being added to the curriculum, or when observation has shown that many teachers teach a particular topic badly.

The radio mathematics project in Nicaragua

The rest of this section will discuss a third type of programme, one in which radio is essentially used to teach the entire mathematics curriculum. At first sight, this may seem an implausible enterprise.

While it is true that a few topics, such as geometry and measurement, offer major obstacles to teaching by radio, it is surprising how much can be done with this apparently limited medium. For some topics, such as mental arithmetic, it is probably the best approach. In most cases, of course, the radio is not really the only medium used. It will be supplemented by assistance from the teacher, by the blackboard, by printed material and by student notebooks. As an example, we will look at the work of the Radio Mathematics Project (RMP) in Nicaragua, beginning with a brief description of the project and then going on to examine its impact on primary-school teachers.

RMP was funded by the United States Agency for International Development (USAID) and the Nicaraguan Ministry of Education. Technical assistance was given by the Institute for Mathematical Studies in the Social Sciences at Stanford University. Complete sets of radio programmes and teacher's guides for mathematics in grades 1 to 4 were produced during the period from 1974 to 1978. The programmes were especially designed for Nicaragua, but, at the same time, and in view of the similarity of primary-school mathematics curricula in most countries, they were made so that they could be adapted for use in other countries. Although the official Nicaraguan curriculum was generally followed, changes were made whenever topics seemed inappropriate. The programmes emphasize traditional mathematics skills, with little 'modern' mathematics being taught; but use was made of recent research in teaching methods. The programmes are oriented for use in rural areas in developing countries, as this is where the greatest shortage of trained teachers seems to be. Achievement tests show that Nicaraguan students in radio classes perform at a much higher level than do those in conventional classes, especially in remote areas. Results in Thailand confirm these findings. RMP programmes are now being adapted for use in several countries.

The RMP programmes are presented as dialogues between the radio teacher and the classroom children and they seek to elicit frequent responses from the children. With careful timing, it is possible to make children feel that the radio teacher is listening to them. Student responses occur at the rate of four to five per minute. They may be oral (group), written (individual) or physical (games and concrete materials). The programmes are popular with children, owing to the mixture of active response and the use of many entertaining items, including songs, games and puzzles. An important feature is the variety of the items within each broadcast. Several topics are usually treated in a single lesson. This is possible because the approach makes use of 'distributed practice'. Topics are not taught in a block; they are spread throughout the year. In anticipation of irregular attendance in rural schools, much repetition is built into the sequence. The repeated items are similar rather than identical. A complete description of the RMP methods and programmes is found in Friend et al. (1980).

Each daily lesson consists of a radio broadcast, usually lasting about 25 minutes, and a 15-minute follow-up led by the teacher. For this, the teacher may use the guide provided by RMP. Most of the instruction is given during the broadcasts. The follow-up serves two main purposes: to cover topics, such as geometry, which are difficult to teach by radio and to give additional practice on other topics. The broadcasts cover the entire curriculum. No textbooks are necessary. Owing to the expense and the practical difficulties of conducting an extensive teacher-training course, the programmes are designed for use with a minimum of special training. A single 3-hour lesson with the teachers is sufficient.

The format of RMP programmes varies according to the grade. This has different implications for the role of the teacher. In first grade, the teacher distributes worksheets to each pupil and supervises their use during the broadcast, as well as in the follow-up. During the broadcast, the teacher concentrates on helping individual students while in the follow-up, worksheets are used much as in a traditional lesson.

No worksheets are used in the second grade. The teacher must write material (given in the teacher's guide) on the board, before the broadcast begins. Some of this is copied by students into their notebooks; some will be used in group activities. During the broadcast, the teacher's role is similar to that in first grade, except that attention is focused on the board and on student notebooks.

In grades 3 and 4, the role of the teacher is quite different. During the broadcast, the students copy into their notebooks problems dictated by the radio teacher. The follow-up is strictly supplementary, and the broadcasts can easily be followed without participating in the follow-up. This approach allows for the fact that in many rural primary schools the teacher must handle more than one grade. The broadcasts can therefore leave the teacher free to work with children in another grade, if this is necessary. If the teacher has only one grade, then the broadcasts provide a good opportunity to work with individual students.

The role of the teacher

Evidently, RMP defines a role for the teacher that is rather different from what he normally plays. The teacher, as before, must perform all the usual functions of supervision, maintenance, discipline, administration and student evaluation. But now the teacher has time to help individual students and so becomes more of a tutor. The teacher can observe individual students as they are working and help those who are having difficulties at the moment that they need help. If many students are having difficulties with a topic, then the teacher can deal with them after the broadcast.

Evaluation of RMP

RMP, like any well-designed radio programme, is expected to serve as a model of good teaching practice. This is as it should be. There is anecdotal evidence from Nicaragua that RMP also improved the methods used in teaching other courses. Some of the teachers also said that the broadcasts helped them to learn mathematics along with their students. In general, analysis of teacher questionnaires in both Nicaragua and Thailand, as well as of comments by many observers, indicates an overwhelming acceptance of the radio mathematics programmes by the teachers.

Summary

In this chapter, some of the problems faced by primary-school teachers, particularly those working in developing countries, have been considered, often, the teachers are underqualified and overworked. Their schools are poorly equipped. Their classes are overcrowded. A shortage of resources hinders their receiving formal in-service training. Two rather different strategies for helping these teachers, which take their difficulties into account, were presented. While there have been many projects designed to retrain and up-grade primary-school teachers by means of officially sponsored schemes, these efforts should be supplemented by unofficial mechanisms. Subject associations with their meetings, conferences and publications can develop teachers' professionalism and self-sufficiency. Radio broadcasts can put new resources at their disposal, and also, as a by-product, improve their knowledge of mathematics and their teaching methods. Both mechanisms are very cost-effective and deserve wider recognition and support.

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P. K. Srinivasan

Cultivation of climate through mathematics clubs

Introduction

Mathematics clubs, as compared to science clubs, are still a novelty today in schools and in teacher-training colleges. The potential of mathematics clubs for motivating the learning of mathematics with greater enjoyment and less anxiety, for creating in learners a taste for mathematical thinking, for aiding the emergence of talent in the gifted, and for educating parents and people at large so as to enlist their support and patronage (Mmari, 1980), has yet to be tapped in full. Moreover, mathematics clubs can even help to prepare the climate for effecting changes in the school mathematics curriculum, in the methodology of teaching and in programmes of evaluation.

Where they exist, mathematics clubs tend to flourish in post-secondary undergraduate institutions, where mathematics is offered as a major subject of study. Some secondary schools also have mathematics clubs, particularly those schools which have a few enthusiastic and devoted mathematics teachers on the staff. Mathematics clubs are not usually found in teacher-training colleges and in primary schools. With the fast-growing importance and influence of mathematics in the present-day world, there is an urgent need to encourage their growth in teacher-training institutions and in primary schools. This is particularly so in the developing countries of Africa, Asia and Latin America. Where mathematics clubs in teacher-training colleges do exist, their influence upon students, teacher-educators and the community is considerable. Some students have even been inspired by them to organize, as part of their teaching practice, a pupils' mathematics club or to engage pupils in mathematical exposition. Through his involvement in a mathematics club, an interested and devoted teacher can even bridge the gap between the intended curriculum and implemented curriculum. He can thus expand the scope of examinations to include practical work in mathematics similar to the long-accepted practical work in science.

Organization

The organization of a mathematics club in a teacher-training institution should permit both ordinary membership, drawn from students who take mathematics as part of their course work, and the associate membership of students who do not.

Management of the affairs of the club should be entrusted to an elected body of officers drawn from the membership. This may comprise a president, a vice-president, a general secretary, a financial secretary, a treasurer, a social secretary and a publicity secretary. There should be a written constitution. This might make provision for co-opting other members with responsibilities, perhaps, for running a journal or a bulletin, maintaining a bulletin board, putting up displays in a show-case, etc. The posts of president, financial secretary and social secretary would normally be filled from the ranks of final-year students. Other posts should go to more junior students, thereby ensuring stability and the continuity of the club as well as an equitable representation of members.

In primary school, the organization would normally be less elaborate. It might well be sufficient to elect or to nominate just three officers to run the club, such as a president, secretary and treasurer.

It would, of course, be prudent to assign to a mathematics club one or more members of staff to offer guidance on and supervision of programmes, and to ensure the proper management of funds in cases where funds are raised by membership fees or by donation from the student body.

As the name of a club is important in developing prestige and popularity, it is advisable to associate with it the name of some creative mathematician such as Newton, Gauss, Euler, etc. This would remind students of the contributions great mathematicians have made to the growing edifice of mathematics.

Programmes and activities

A mathematics club should provide in its constitution for a well-designed annual programme of activities if it is to be influential in effecting attitudinal changes and evoking and nurturing the talents of gifted students. The programme may consist of periodical meetings, the issue of weekly bulletins, the organization of a mathematics week involving such activities as a students' symposium, a quiz, a 'make and take home models' workshop, learning activities and/or recreational mathematics expositions, mathematician masquerades to highlight the contribution of great mathematicians or to celebrate the birthdays of mathematicians, excursions, a mathematics fair, a mathematics education contest, some small-scale research and brain storming sessions for non-routine problem-solving. The objective would be to maximize the participation of members (Srinivasan, 1981).

There are also worthwhile and highly beneficial activities of a more ambitious type. For example, a number of institutions can agree to join together, to pool and share their resources so as to organize, on a co-operative basis, a screened, rolling relay mathematics exposition, contest or fair open to the public. 'Screened' means making sure that the concepts in the curriculum are covered, but not repeated in the course of a number of years. 'Rolling relay' simply means that the exposition 'rolls' from one institution to another, while it 'relays' in the same institution. A certain number of consecutive week-ends is set apart to ensure that normal work is not disturbed, to reduce the strain of holding the exposition for a number of days in the same institution, to whet social expectancy, to raise the standards of the exposition through public and professional criticism and comment, and to provide scope for the participation of numerous students. Incidentally, such activity can generate interest in launching mathematics clubs, in other institutions. It can provide adult education. Another activity designed to satisfy the needs of mathematically inclined students who seek self-education is an 'exhibition course' in a topic outside the curriculum. Such a course, would, as the name implies, result in an exhibition of some kind. The course could also be rounded off with a test, an evaluation of responses and the award of prizes and certificates.

While examinations tend, in these days of mass education, to 'level down' standards of attainment, expositions can 'level up' standards of understanding through healthy emulation, challenge and concern.

Primary schools

A mathematics club can enhance enormously the activities of a primary school. It is true that a primary-school teacher is not a specialist in mathematics. But the children's attitude towards mathematics is largely formed in their primary-school years. If children miss the excitement and enjoyment that should characterize their experience of learning mathematics, they easily develop negative attitudes towards mathematics, and are anxious to give it up at the very first opportunity that presents itself to them. A mathematics club can contribute a sense of confidence and help children to develop their mathematical gifts and tastes.

The difficulty is to find a primary-school teacher with enough confidence to organize a mathematics club for children. Yet, given administrative support and good guidance, a primary-school teacher can make a success of such an enterprise. And for most primary-school teachers, guidance and backing from above are essential to make up for an inadequate background and to compensate for any mathematical inadequencies there may be. Support can also be provided by the mathematics club in a teacher-training college. It can act as a resource centre for primary-school mathematics clubs in the vicinity; it can be the bank for lesson plans and for teaching aids and so cater to the needs of trained teachers and teachers under training.

Experience has shown that there are many activities that can appropriately be built into the programme of a primary-school mathematics club. Here are just a few of them:

Making a collection of 'stories' that could account for a given numerical statement.

Making a collection of 'stories' that could give rise to a mathematical sentence; the sentence can be an equation or an inequality.

Number composition from a given lattice of points in some geometric shape such as a rectangle, a square, and combinations of such shapes.

Making mathematical statements by placing in alignment rectangular strips with measures indicated partially or fully on them.

Discovering patterns of numbers and patterns of shapes from formulae, or from extending a sequence.

Games based on 'think of a number'.

Creating algebraic expressions from language patterns.

Making easy generalizations and extensions.

Making appropriate definitions.

Reasoning from axioms, and using a counter example to create a contradiction.

Building magic squares, magic triangles, magic crosses, magic hexagons and magic circles.

Using graphs as a model of behaviour, and interpreting graphs to forecast behaviour.

Abstraction from situations which have a common mathematical core of thought, but which seem to be dissimilar.

Demonstration talks on specific topics of the curriculum.

Mathematics without a chalk board.

Simple problem-solving sessions based on problems chosen for their interest, but suited to the background and ability of the children.

Mathematics in the environment—calendar, clock, tiling, textile designs, braids, knots, seating, serializing, shipping, tailoring.

Howlers, fallacies and paradoxes drawn mostly from primary students' responses.

The mathematics club in a teacher-training college can help primary schools to introduce these activities through visits by student teachers. It can also organize, from time to time, inter-primary school mathematics fairs or contests. Such contests can stimulate interest in and knowledge of 'new' techniques. For example, teaching aids are not yet thought to be indispensable to learning by mathematics teachers, and they are not used as widely as they should be. So by holding contests for primary-school teachers in techniques of improvisation in teaching mathematics, using waste materials like empty containers, paper, cardboard boxes, bottle tops, broom sticks, square-ruled sheets, etc. is encouraged. Contests for primary-school teachers can also be based on ingenuity in the treatment of a particular topic, and on the possibility of teaching a topic in the syllabus of a higher class to a lower class.

Bulletins can regularly be sent to schools, and feedback can help to improve the bulletin. A bulletin board can carry a quiz, with a 'discovery box' to collect pupils' responses. These can make it possible to detect and foster mathematical talent among primary-school pupils.

Caveat

A mathematics club should not be allowed to degenerate into an examination-coaching centre or special-classes centre. The tutor assigned to guide club activities should do all he can to provide the members with opportunities to enjoy mathematics. The ethos of the club should be conducive to self-reliance and self-confidence, so as to facilitate the mathematical growth of members. Tutoring should be avoided. Members should be encouraged to feel free to commit mistakes and learn from them. The tendency to consider the club merely as an opening for distinguished people to be invited to give an occasional talk should be avoided. A club that does not plan for and secure the large-scale participation of its members can scarcely justify its existence.

Support

The inspectorates should be alive to the importance of mathematics clubs. They should maintain a directory of them, and gather reports on their activities. When inspecting the schools and colleges of their zones or districts, they should make a point of recording the existence and work of associated mathematics clubs.

Each school or college can maintain a register of interested persons, drawn from professors, lecturers, engineers, scientists, doctors, etc. in the neighbourhood or in neighbouring institutions or higher learning.

Schools and colleges should also expect to be able to look to the national mathematics association in a country for guidance and support.

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Guy Brousseau

The IREM's role in helping elementary-school teachers

Introduction

In the late 1960s, it was becoming apparent that mathematics was likely to play a central role in science and technology. However, the inadequate teaching of mathematics placed a barrier to communication between the scientific community and the rest of humanity. Mathematics was even considered by some to be an instrument of intellectual segregation, or at least a significant handicap to the development of certain countries. A better knowledge of mathematics had, therefore, to be propagated and made more accessible to the majority of people. The attitude of society towards mathematics had to be improved in order to foster scientific development, to promote a more equitable sharing of responsibility for its consequences and to make it more humane as well as more efficient. And, in order to reduce the gaps between the form in which mathematical knowledge is displayed in general cultural life and that in which it is used in the scientific world, modern ways of presenting the discipline were needed. Such were the reasons that were advanced for giving mathematicians the means of determining where updating was needed in order to improve and co-ordinate the teaching of their discipline and exercise an 'epistemological supervision' over it.

The *Instituts de recherche sur l'enseignement des mathématiques* (IREMs), which began to be established in France in 1969, were specifically designed with this in mind. At present, there are twenty-five IREMs. They seek, above all, to serve as a place for meetings and exchanges among all people who are concerned, in one way or another, with the problems of mathematics education, regardless of level. With this in view, the institutes were entrusted with the task of providing in-service training. They then requested the means to undertake research into teaching in order to obtain fresh material for their studies. They also wanted to participate in discussions on syllabuses, to help in the work of initial training and produce curricula designed to promote the solutions they recommended. However, apart from in-service training, these tasks had already been assigned to other bodies, which added them to responsibilities in fields sometimes far removed from each other. Thus, it was also necessary to assist and guide these bodies, which had hardly any means to devote to the search for new solutions, or to their co-ordination; this had to be done without supplanting them.

Unlike the traditional French education structures, which are highly centralized and respect a rigid hierarchy, IREMs are independent and regional university bodies. Each has its own chosen lines of approach and organizes its work according to priorities perceived by its own teams. Some activities, such as those concerned with training, follow a fairly uniform pattern. Others, such as research or production, are conducted on a basis of association or complementarity. It is therefore difficult to define a standard model for IREMs, particularly since the listing and classification of all the documents produced have proved to be virtually impossible, as will be explained below.

About fifteen IREMs have done mathematics work in the field of elementary teaching. A special issue of *Bulletin inter IREM* (No. 16, 1978), prepared by the standing IREM committee for elementary teaching (COPIRELEM), has reviewed these activities.

To illustrate the various roles that an IREM can play in support of primary-school teachers in the matter of mathematics education, we shall examine in the following pages a precise case, namely the Bordeaux IREM, where this kind of activity has been consistently pursued, since the very beginning, around a clearly defined long-term project. Other original activities undertaken in other IREMs will also be mentioned, when appropriate, in the hope that we shall thereby give a clearer idea of the coherence, needs and difficulties of the venture. But it must be understood that only a few IREMs have adhered to the project and have undertaken comparable activities.

Motives and purposes

In the late 1950s, elementary-school teachers had no reason to suppose that it was necessary to reform the teaching of arithmetic. Its purpose was to integrate most of the pupils into the world of work at the end of compulsory schooling. It had its own methods, its specific language and its evaluation techniques. The small batch of children (approximately 25 per cent) who went on to a higher level of education would then begin to study mathematics. They would be expected to have been introduced only to 'elementary algorithms' at primary school.

As a consequence of the prolongation of compulsory schooling and the determination to put an end to the system of 'two cultures', it eventually became apparent that, in fact, the teaching of mathematics was weak at two crucial points: at the primary-school level (ages 6 to 11) and at the lower secondary-school level (ages 11 to 15).

Consequently, the main objective became to make complete courses accessible to all and to offer vocational training only as pupils are severed, at different stages, from the trunk of a 'common core' curriculum. This well-intentioned ambition implied risking high stakes on the possibilities for the unification and step-by-step presentation of knowledge.

In mathematics, the wager did not seem too risky. In the first place, there existed a huge body of reorganized knowledge incorporating the findings of research into basic mathematics that had been carried out over the past hundred years. And, in the second place, there were the hopes nurtured by the structuralist standpoint, with the support of the findings of research such as that done by Piaget in genetic epistemology.

To simplify, it can be said that previously, at the primary-school level, 'basic mechanisms' were taught. These were to be used later, as such, within the framework of another way of formulating and explaining notions, so that it was permissible, up to a point, for understanding to come only after the learning had been done. Henceforth, the pupils were to start by grasping basic, general concepts. Afterwards, they could be introduced to the arithmetic operations.

It soon became clear that the emphasis should be laid on the meaning of the notions learned. These had to be correctly grasped from the outset. Attention, accordingly, switched to the pupils' ability to understand, formulate and explain what they were taught. This provoked the need to conceive new teaching situations so designed as to organize interactions between the pupils and their environment and arouse in them the desire to appropriate and to use mathematical concepts. When the IREM undertook to satisfy this need, it was found that such interactions were specific to the kind of knowledge produced and therefore varied from one notion to the other. Nevertheless, it seemed reasonable to think that the axiomatization of mathematics would enable the necessary unification and economy measures to be taken.

However, IREM staff members who worked on elementary-school teaching were not content to aim simply at raising the level of the small group of pupils who were likely to carry their scientific education further, nor even to bring about an improvement in the quantity of mathematics, properly so-called, which all children are taught during their years of compulsory schooling. Their main purposes were to cover a wider educational spectrum.

In primary school, the pupils begin to discern different sorts of reasons for believing that a statement is true: ethical, aesthetic or logical. At that level, the mathematics class is the best place (though it is not the only one) in which to be introduced to the management of truth and rationality.

The children should learn not only a technique for reasoning, but also a social skill: how to convince an opponent without showing disrespect and how to bow to the force of his 'arguments' without losing one's integrity, without being influenced by personal charm, rhetoric, injured pride, authority and force. This means understanding how and why the responsibility for what is said in one's environment is being constructed.

If the subject and his desire do not slip into their place, nor express themselves, nor construct themselves in these discussions, then a good relationship with knowledge cannot be established. Therefore, it is clearly important, while putting the emphasis on the acquisition of individual skills, to seek at the same time, by presenting certain types of situation to the children, to build up an example of the social relations that govern the production, management and communication of knowledge.

These intentions reflected very general and very profound educational purposes, ranging from the promotion of good citizenship to the full development of personality, from training in the priorities of democracy to the elaboration of the cognitive subject, and these extensions were described clearly and explicitly as early as 1968 at a symposium in Amiens.

Objectives

The objectives of the IREMs, as far as primary education is concerned, can be deduced from the above considerations. What has to be done is to prepare teacher educators, counsellors and officials for the change in basic objectives, and then to assist them in putting the change into effect. The first training objective, then, was to educate a large proportion of the above-mentioned personnel not only in the new organization of mathematical knowledge but often in entirely new contents as well.

But it was already clear to many people at the time, and it was confirmed later, that it would not be enough to teach mathematics to teachers and to teacher educators to ensure their ability to create appropriate teaching situations. It was also necessary to work out, together with teacher educators, a minimum set of teaching materials, to be used by teachers after quite a short course of training. These materials were intended to enable them at least to understand the new notions and to help them, if possible, to solve teaching problems that the designers might have overlooked. However, this combination of educational contents and examples was likewise found to be inadequate. If the principles of mathematics education were to be understood and mastered, they had to be treated as a new scientific field, and certain specific concepts had to be created. It followed from this observation that it was necessary to establish, in addition to certain research objectives, which will be discussed below, a third training objective, namely to teach the basic methods of organizing and supervising teaching situations, together with the relevant theoretical concepts.

The three above-mentioned types of objectives emerged only very gradually. They were not accepted at the same time by all the IREMs. This was because the very pragmatic approach adopted by the institutes led them to raise the problems in very different ways. These will be examined below, along with the research objectives.

In the late 1960s, proposals had been made and many different types of action had been taken that had resulted in new primary-school curricula. Because they made room for basic mathematical concepts, these curricula gave teachers a motive for taking an interest in the new ideas and justified intervention by IREMs. But even the most concrete of these proposals still gave rise to many questions and difficulties. It therefore became necessary to set research objectives which can be described as objectives for 'applied and developmental research'. This involved evaluating current practices, progressively producing new suggestions and more advanced teaching aids and materials in order to ensure a steady supply of feedback, which is essential to the health of a training programme. This objective was retained even when the in-service training of teachers was assumed by teacher-training colleges.

At first, the solutions considered were mainly pragmatic and empirical. It was believed to be sufficient to talk a problem over together, to contact various specialists, consulting each individually, to find the time to try out the selected solution and then to find the means of disseminating and explaining it. Teachers were thought to be in the best position to adapt it and evaluate its results, and it was accordingly assumed that they alone would be required to do so. That is why research undertaken with a view to producing means of acting on teaching had, of course, to be closely associated with the retraining of teacher educators and the training of teachers. The partners, whatever their background, found IREM a place for discussion, where hierarchical constraints could be temporarily set aside, and they could work in a spirit of co-operation, personal commitment and friendship. This type of activity, sometimes called 'action research' is undoubtedly useful. However, it was not long before its limitations became apparent.

First, the changes that can be regarded as feasible are greatly limited by the teachers' previously acquired knowledge and their habits. It is therefore necessary to plan time-phased activities and even to produce, as is done at the Bordeaux IREM, different brochures to suit the varying amounts of information possessed by the teachers, or their pedagogical choices. Second, the fact that the only evaluation criteria for methods are either *a priori*, determined by the author, or, *a posteriori*, arising from the success of the results among the users, finally prevents any real discussion from taking place. The argument of 'novelty' therefore overrides an inquiry into the value of the methods, which succeed each other as fashion dictates, obliterating or ignoring their predecessors.

Even if a more conventional idea of applied and developmental research had been accepted, other serious problems would still have remained unsolved. These relate to the choice of teaching methods and the justification of such choices, their reproducibility, their communicability to teachers and, in general, the supervision of activities and results. It was then thought that the solution to these problems depended, first of all, on a better knowledge of the basic phenomena of didactics. Certain IREMs, therefore, began to develop basic research, with the intention of contributing in the near or more distant future to the supervision and orientation of activities concerned with training and action-research. These basic research projects can be divided into two major categories according to their conceptions of the antagonistic and complementary roles played by theory, action and experience in didactics. In the first category, the research seeks to give even more precise and detailed descriptions of the teaching system as it stands and as it is evolving—what the pupils know and how they behave, how the teachers behave, etc. These investigations are carried out by such means as surveys, achievement tests, questionnaires, clinical interview and 'genetic' studies. In the second category, the research seeks to reduce the discrepancies apparent in the functioning and the results of teaching processes and situations. The study of the reproducibility of teaching methods progressively led the experimenters to the conclusion that observation should be conducted by numerous teams composed of various specialists. New concepts had soon to be created to describe the didactic situations, to identify, list and classify in order of importance, the significant determining conditions. It then became possible, in certain cases, to make real forecasts of the results of a change in teaching conditions. These analyses led, in turn, to the identification of real didactic phenomena, which are now being actively studied, and the emergence of a sort of didactic engineering, which uses this theoretical knowledge as guidance for the systematic production of situations or methods whose results can be forecast to some extent.

This set of objectives, which was published by the Bordeaux IREM as soon as it was set up and has been resolutely pursued ever since, has not met with unanimous approval. In fact, it is the lack of agreement on the use of didactic concepts that made it almost impossible, as yet, to identify and classify the texts produced in IREMs.

Secondary objectives

The first results of the research were encouraging, but they gave reason to think that if the teaching of certain notions was to be appreciably improved, many conditions would probably have to be changed, simultaneously and quite radically, and that it would accordingly be necessary to make great efforts in various directions. An in-depth study, conducted from 1967 to 1973, of the prerequisites for an experiment in the teaching of mathematics brought to light the need for the creation of an observation centre in a consolidated school. Two centres of this type were established, one of which is still in operation. This centre, which we shall discuss below, makes it possible to produce and observe considerable modifications in teaching conditions without incurring risks for the children or the administration. A centre of this kind has become such an indispensable means of attaining some degree of efficiency and credibility that its operation has been one of the most important objectives of the Bordeaux IREM for the last ten years.

Research of a methodological nature is also essential; for example, the interdependence and interconnection between teaching conditions and behaviour patterns are the subject of profitable studies (Pluvinage, 1977; Gras, 1979), some of which are based on the results of the experiments and draw upon the concrete problems encountered during those experiments. A great effort of analysis has made it possible to define and compare situations, list contributions, classify the questions raised and discuss the validity and originality of the theories advanced and of the methods of proof. We are thus witnessing the emergence of a field of knowledge concerned with the part of educational activity that is specific to what is being taught and which is defined in France as 'the didactics of mathematics'. Besides theoretical knowledge, this field comprises practical knowledge based upon what is needed in order to produce and supervise teaching situations, materials and equipment. Research into the didactics of mathematics is developing within a community of researchers, which has organized its work, over the last five years, in such a way as to enable a scientific debate to be pursued through the usual institutional channels. This research has gone far beyond the initial framework of the IREMs, since bodies coming under the co-ordinated research programme of the Centre national de la recherche scientifique (CNRS) are now involved, such as the Centre d'études des processus cognitifs et du language [centre for Studies of Cognitive Processes and Language], IMAG of Grenoble and the Institut national de recherche pédagogique (INRP). But IREMs have contributed a great deal to this development, and among those that have made a considerable effort for elementary education, the most outstanding are those of Paris, Grenoble and Bordeaux (which has been entitled, since 1975, to award *diplômes du 3e cycle* in the didactics of mathematics).

Interdependence of objectives

We have attempted to show that all the objectives were complementary and interdependent, like the measures taken to achieve them. The need to train inspectors, for example, stimulates research and the formulation of findings. The demands of the experimental method are conducive to the production of teaching situations that are well thought out, reproducible and satisfactory from all points of view. The effort to describe and observe, itself motivated by the resolve to make the research the subject of scientific debate, enables unforeseen factors in the teaching situation to be identified. The activities mentioned in the following paragraph were progressively developed, as a result of such motivation, on the model of 'spiral' development, progress along each axis being conditioned by the progress made in the neighbouring sectors. Furthermore, some activities came to an end and others were initiated as needs and opportunities arose. The following historical account will therefore be slightly distorted, as the research teams were not all doing the same thing at the same time.

Prerequisites for IREM activities in primary schools

The organization, in 1970, of in-service training for primary-school teachers in teacher-training colleges was the last step in the establishment of a very complete system, officially entrusted with the task of providing primary-school teachers with all necessary forms of assistance, namely:

The pre-service and in-service training given in teacher-training colleges and in the teaching districts, with the collaboration of the teaching staff of the training colleges, the school inspectors and education counsellors.

The documentation and promotion of 'action research' in the regional centres for educational documentation and in the Equipe départementale de recherche et d'action (EDRAP).

The promotion of various forms of educational research organized and co-ordinated mainly by INRP through the regional or departmental committees.

The dissemination of teaching aids by the Office francais des techniques modernes d'éducation (OFRATEME) or by private publishers.

Thus, in principle, IREMs are not invested with the administrative responsibility for giving direct assistance to primary-school teachers. Nor are they invested with full scientific responsibility, for it is clear that, at this level, considerations of a psycho-pedagogical nature and the aims of the overall education plan outweigh the demands of the discipline. Therefore, the mathematical knowledge that IREMs can provide will have to be made into lessons for the pupils by a long process of step-by-step adaptation that only the education system is thought to be able (and authorized) to carry out, supervise and follow up. In fact, this system does not work very well. Among the symptoms and causes of malfunctioning, most of the following have been repeatedly criticized:

As no examples of lessons are given, most teachers cannot teach the new contents satisfactorily.

Hence the need for—and existence of—a profusion of 'research projects' and experiments aiming mainly at innovation and the spread of innovation by every possible means. These research projects and experiments are conducted at all levels of the system and are impossible to co-ordinate or compare. For, while they often copy each other, they do not seek mutual support, nor do they ever voice direct criticism of each other; on the contrary, they protect themselves by invoking authorities of all kinds (including the money-making tradition that sanctions criticisms of literature, but not of school textbooks).

The transition to the development phase leads to cruel disappointments, since the application of the 'innovations' is not guided by knowledge of a scientific nature; the prerequisites for reproducibility are unknown.

The teacher educators find it difficult to define the content of their teaching: apart from the basic disciplines—mathematics, psychology, pedagogy—everything that is necessary to the organization of didactic activity is regarded merely as deserving a few comments and is thought to baffle any attempt at theorization, conceptualization or enumeration.

No one can bring together all the means to make an overall study of a teaching operation, including the choice of a project, the working out of a method in the light of a scientific knowledge of the learning and teaching processes at stake, the training of the teachers, the supervision of the operation and the evaluation of the results.

The chief cause of these difficulties lies in the absence of real feedback for a whole set of decisions, including those concerning the didactics of mathematics: the teaching is not getting better, being improved neither by minor rule-of-thumb alterations, nor by any other method; inertia is its sole protection against a tendency to slide backwards. Having been designed on the administrative model, this system, which is organized for the transmission of knowledge, makes no provision for the institutionalization of such new knowledge as may come to light during its operation, and this is particularly true of knowledge about teaching itself. On the contrary, the system is opposed to doing anything of the kind. The curricula, instructions and further training courses decided on by the administrative department are applied in obedience to orders issued as if to subordinates, the evaluations are investigations and the research bodies are called research units. In the final analysis, orders and requests for clarification are the only messages compatible with this administrative approach. Reflection is officially encouraged, but remains a private and gratuitous activity. From every point of view, the business of teaching is totally centralized and rigidly compartmentalized, which makes mutual consultation very difficult: within an organization of this kind, the official admission that a difficulty had arisen would be bound to lead to the formal appointment of a special problem-solver. Thus, training and research have an ambiguous status in such a system.

Action and methods of action

Principles

On the basis of the analysis presented above in a very condensed form, IREMs have sought to remedy defects and to perform the co-ordinating roles that no one had assumed, so as to help the system to work better, while applying, as far as possible, some feedback techniques. This led them to take part in all types of teaching activities comprising a mathematical component, in order to study them with teams that were more varied, larger and better centred on an overall project than those that could be brought together by the various bodies so far involved. These studies produced, and are still producing, many suggestions based upon thorough analyses and genuine research.

The results are brought regularly to the attention of the responsible authorities, but are not forced upon them. Rather, it is left to the latter to follow them up, or to decide whether to apply them or not. For this purpose, IREMs organize many activities (workshops, meetings to agree on concerted action, lectures, etc.) which participants attend only because they feel that they will learn something useful. In order to keep up the pressure discreetly, IREMs must retain the means of disseminating their acquired knowledge through other channels, should this become necessary. Similarly, they must ensure that they cannot be cut off from sources of information and must be able to conduct regular surveys on large groups of pupils.

In order to perform this dual role of partner in dialogue and counterpower tactfully and efficiently, IREMs must maintain close relations with elementary education and become thoroughly familiar with it, which pre-supposes that co-operation is forthcoming from various sources.

Forms

IREMs' activities concerned with primary-school teachers have taken rather varied forms:

As regards targets: pupils, parents, serving or trainee teachers, teacher educators, administrative authorities, school inspectors and education counsellors, innovators, research workers and master teachers.

As regards means: mimeographed texts, articles in various types of periodicals, books, films, lectures, workshops, debates and mathematical clubs.

As regards types of exchanges: promotional texts, documentation (supplied on request), information (supplied on the initiative of the Institute), training (negotiated).

As regards contents: mathematics classes, classes on didactics, texts on history or epistemology, the description of a series of teaching situations, reports of observations, studies of objectives, comments on curricula, findings of research or of surveys, experiments to be carried out, etc.

As regards channels: its own channels, privately-owned channels, OFRATEME, CRDP, RTS (Radio-télévision scolaire, or school broadcasting), etc.

Every one of the methods used by each of these forms and types of activity has been tried out or experimented with at least once by the Bordeaux IREM. Their range can be judged from the impressive number of documents produced. Relatively few copies were made of each document because the action in question was indirect. But the continuity of the effort is attested to by the number of documents dealing successively with the same subject, for this is tackled by different teams which criticize and recapitulate their conclusions.

The cognitive content of the activities

Review of the mathematical subjects dealt with shows that they cover all the topics in the elementary-level course. The public has been impressed in particular by innovations in the teaching of logic at an early stage with the help of diagrams or Dienes multi-based logic blocks. Little has been published on these matters, however, and most of the few publications that do exist tend to criticize the simplistic applications proposed in earlier textbooks and the structuralist theories underlying them. On the other hand, the studies of methods of teaching computation, operations in which natural or decimal numbers are used and problems of addition, multiplication, order relations, geometry, probability and measurement are full of original suggestions. They all stress the importance of devising situations in which children can understand, create and put into operation the concepts they are being taught. Many situations of this kind are to be found in the 1978–80 programmes.

By classifying these learning or teaching situations and processes according to whether they involve certain relationships or not, it has been possible to describe and interpret more accurately the behaviour of teachers and pupils and hence to foresee certain effects or account for differences in the results achieved by what appeared to be similar lessons. The situations may fall under the headings of action, formulation, proof and institutionalization (or removal from context) of knowledge. This method of classification puts forward a more or less specific hierarchy of mathematical criteria, or of particular concepts in that field, that should be studied in order to control the production, reproduction and analysis of such situations. These criteria make it possible to appraise, for instance, the degree of anticipation required of the subject and the risks he incurs, the informational leap implied by the situation, communication constraints or constraints imposed by the teaching contract and their effect on the meaning of observable manifestations of knowledge, etc.

The possibility of exercising more effective control over the organization and the progress of classes from both the theoretical and experimental standpoints made it easier to bring to light and then study certain basic phenomena of the teaching process, such as didactic transposition, various kinds of obstacles and failures, etc. The main effort of research has gone into reopening the question of what is usually referred to as the learning of 'mechanisms'. What is the source of the idea that the conception, application and understanding of notions operate separately and in different ways according to whether they are being invented, used or learnt? Is each user of mathematics doomed to be an actor, imprisoned by a text which has been written elsewhere and which he finally has no choice but to recite? How has it come about that conditioning, in a field where it is found to be so alien to the spirit and practice of mathematics, is still seen as a binding obligation (both by the man in the street and by a certain Nobel prize-winner)? In this field, useful arguments have been provided by the history and epistemology of mathematics.

In the final analysis, the essential element of the IREMs' activities, and the justification of their existence, is in their extremely relevant, original and responsible research work. It is not possible to describe that work here. But it should be pointed out that the best way of helping teachers is not merely to cater to the needs expressed by themselves, but to encourage them to look beyond the necessarily limited analyses which they make in the light of their own professional experience, and to start by proposing explanations, suggestions or solutions, then to try to protect part of their activity from the tyranny of ideologies and fashions and to show that while some phenomena can be controlled, others cannot, and that while some things can be expected of them, other demands are contradictory.

In certain countries, the need for a close link between the training of teachers and educational research is recognized. All too often, however, this link has neglected the specific items of knowledge to be taught. In the United States, for instance, a formidable amount of research on didactics is being done. But, with a few exceptions, the term covers a fundamentally different activity in that neither the subjects studied nor the methods used are the same. There are signs, however, that the situation is improving, and that the epistemological component of didactics is gradually winning recognition.

Conclusions

To sum up, IREMs can justify their activities on the grounds of competence in two fields: first, in mathematics, and second, but more importantly, in didactics, this competence having been confered by the theoretical and experimental research into teaching processes they have been able to conduct in the field or in their own observation centres. Their activities are worthwhile in so far as their proposals are submitted, and recognized, as ways of improving children's performance in the context of a global education project. They are based on discussion, negotiation, the search for truth and for conclusions that are set out clearly for everyone to scrutinize, and on rigorous criticism and querying. But they tend more to supply explanations for errors or difficulties than to pass judgement, and they attach greater importance to understanding phenomena than to evaluating them or meeting the demands of action. For these reasons, IREMs must refuse to be incorporated into the decision-making system.

Favourable factors

However, the decisive factor that has enabled the Bordeaux IREM to succeed, and to persevere in certain activities that were either not carried out at all or ran aground in other IREMs, has been the existence, from the outset, of a team of organizers. This team has comprised about fifty mathematics teachers from teacher-training colleges in the region (some of whom have been making a weekly journey of 300 kilometres for over ten years in order to give their assistance). They, in turn, have enjoyed the support of specialists working in various branches of higher education (two mathematicians, one physicist, two educational psychologists, one psycholinguist, two linguists), together with primary-level teachers and officials concerned with education. As this team made progress, it took in new members. It was welded together by an ambitious, original and open-ended programme of research and activities designed to be useful to everybody.

A second factor was the establishment of an observation centre attached to the Jules Michelet de Talence consolidated school. In this centre, for which the IREM has assumed the scientific responsibility, twenty-three teachers spend two-thirds of their time working in ten elementary classes and four nursery classes. Each teacher spends one-third of his time organizing the work of observation. In this way, the members of small teams can work together closely enough and long enough to make and study considerable modifications in teaching conditions without incurring risks for the pupils within the centre. The IREM's activity consists in:

Keeping pupils and teachers under constant observation and gathering longitudinal data over a long period so as to bring to light phenomena or processes, whether induced or not (for instance, the obsolescence of teaching situations or of processes, and reactions to evaluation). This has enabled techniques to be devised for managing pedagogical data that can be used in experimental classes to facilitate the initial training of teachers.

Organizing teaching situations designed to enable the pupils to carry out didactic observations or experiments. This means producing series of lessons for the teaching of a mathematical concept. In most cases, they are new lessons and the prerequisites for their application and reproduction have to be determined.

Preparing experiments or surveys involving larger populations or to be proposed to training or research bodies.

A third factor was the development of good relationships with the users: the training of IDENs [departmental inspectors of the national educational system], PENs [teacher educators], and through them the school-teachers. All were approached in a spirit of equality, responsibility, freedom and mutual respect.

A final factor was the introduction of a *troisième cycle* course in the didactics of mathematics in the university. The production of the knowledge required for the training of teachers, which was made possible by the activity of high-level researchers in the school in question, greatly helped and motivated the teams of organizers.

But perhaps the crucial factor was an ideological one. The researchers have applied themselves unflaggingly to the task of determining the conditions in which children should be placed in order to carry out, formulate and generate activity of a mathematical nature, given that assimilation and learning must be based as far as possible on such use and practice as give a meaning to what has been learned. In many cases, teachers have had little opportunity to engage in this kind of activity in an atmosphere of mutual enrichment and personal commitment. Their only memories of mathematics are of lectures, hidden or revealed certainties, learning, memorization, applications, interrogations, mistakes, fears, and so on, all far removed from genuine mathematical activity. By giving teachers the idea and the means of organizing in their classrooms a kind of club of enthusiasts with a passion for rationality and the process of arriving at a consensus by furnishing logical proof or by tracking down errors, the IREM has revived the mathematics hour, making it a time of true education when instruction in the subject is given in such a way as to develop all the pupils' potentialities.

Difficulties

We have laid a good deal of stress on the institutional difficulties. No training is possible unless there is a specific body of knowledge to impart. This implies research and therefore the availability of researchers and the exploration of new ground. It presupposes as well the existence of theories and methods. The latter can be worked out only in the course of experimental research. This in turn presupposes the establishment of appropriate research centres. But these centres can come into existence only if there is already an established body of knowledge and researchers.

Here it is not so much a question of difficulties as of real epistemological and social obstacles to successful teaching. As long as an obstacle has not been overcome, the first hint of a new solution sets in motion corrective mechanisms. These drive the system back to its previous state. The future will tell whether we were ready to cope with these problems. The fact is that the only sound way of discovering how to make reasonable demands on elementary-school teachers and give them the right kind of assistance in this work is precisely to build up such a body of knowledge.

Results

The IREMs' work on behalf of elementary education has been an acknowledged success with respect to the quality of the educational content proposed, the principles of action, the theories developed and their repercussions on teaching practice. Even so, it is quite likely that this work will be regarded as a failure by both teachers and political authorities. Let us examine the current situation in France and compare it with the stated objectives. At present, there are very few mathematicians who are still convinced that they have some responsibility for the training of teachers and for the epistemological supervision of elementary education. And those who are convinced do not generally consider it necessary to equip themselves with the paraphernalia of minor skills, such as teaching methods, in order to be listened to. Besides, the ministry asks no more of them in this matter than their qualifications in mathematics. In 1980, the training of primary-school teachers was re-organized. The decision to base the training of primary-school teachers in all disciplines on a 'single model' makes the breakthroughs in the didactics of mathematics ineffectual. Moreover, there are by no means enough teacher educators with the requisite knowledge and research experience to take charge of such training even if it were established. Nor do these teacher educators have the right status. Most of them are members of the staff of teacher-training colleges who have taken an interest in this field of knowledge because they have had occasion to appreciate its usefulness.

Research into didactics is itself doomed to naiveté and/or failure, since it is becoming more and more difficult, as the body of knowledge concerned with didactics grows steadily more complicated and technical, for a good student of mathematics to devote enough time to absorbing it before he sets out to break fresh ground in this field. For some people the only hope lies in the ideological field. Can we look forward to seeing the mass of people take a passionate interest in the teaching of mathematics, as was the case from 1965 to 1970? Alas, the courses that have grown out of the research we are discussing here are not disseminated, nor understood, nor taken up. The illusory transparency of didactic facts tends to give everyone the impression that he is capable, without any preparation or previous knowledge, of judging and analysing any statements made about teaching. It also encourages people to think that all that is necessary in order to enable teachers to solve their problems is to give them the material means of purely pragmatic action. This illusion is stronger than ever and never fails to lead to the same mistakes. For example, in the vitally important dialogue between schools and parents, people still seem to believe that these questions can be settled by appealing to authority and dispensing with mediation and appeal channels in the shape of research bodies that would be both truly independent of both schools and parents and alone in a position to provide the general public with useful guidance and information.

For the time being, and for all these reasons at least, the IREMs' work with teachers is difficult to carry out, defend and appraise. It is to be hoped, nevertheless, that the great efforts that have to be made, quite often by unpaid volunteers and sometimes in the teeth of general opinion, or even in defiance of extremely powerful forces, will not have been pursued in vain.

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Biographical notes

GUY BROUSSEAU. Lecturer in mathematics, University of Bordeaux-I, Bordeaux, France. Born: 1933. He worked as a primary-school teacher from 1953 to 1963. Since 1959, he has devoted himself to the study and reform of mathematics teaching at elementary-school level, at first in schools, then from 1965 to 1970 at the Bordeaux CRDP and finally at the University of Bordeaux-I, where he was appointed in 1970 after having completed his higher education. In 1964 he published one of the first works on this question. He helped to set up the IREM and then founded and directed the Observation Centre that is run in association with the Jules Michelet School at Talence. Since 1975 he has been in charge of the teaching of Didactics and Statistics, forming part of the Troisième Cycle course in the Didactics of Mathematics, while still continuing to teach mathematics. He is currently secretary of the International Commission for the Study and Improvement of the Teaching of Mathematics, with whose work he has been connected since 1961.

M. A. (KEN) CLEMENTS. Senior lecturer in mathematics, Monash University, Melbourne, Australia. After completing his undergraduate degree in pure and applied mathematics, he subsequently obtained the M.Ed. and Ph.D. degrees from the University of Melbourne. He taught in schools for ten years before taking up his position at Monash in 1974. His main research interests are the history of mathematics education and cognitive processes, especially imagery, involved in mathematics learning. In 1975 he served as a Unesco consultant in Thailand, and in 1980 he worked in Papua New Guinea, where he investigated factors influencing mathematics learning in that country.

BENJAMIN A. ESHUN. Lecturer in the Department of Science Education, University of Cape Coast, Ghana. He was previously a secondary-school teacher and Chairman of the Mathematical Association of Ghana. He is a member of writing teams for the Advanced Mathematics Project and for the West African Regional Mathematics Programme texts. His chief interests are curriculum development, mathematics teaching and early number concepts in children.

JOSEFINA C. FONACIER. B.Sc. in Education *(cum laude)*, with a mathematics major, at the University of the Philippines, 1948; M.A. mathematics, Columbia University, New York City. Senior Specialist in Mathematics Education and Assistant

Director of the Science Education Centre, University of the Philippines. In addition to mathematics teaching, her previous duties have included in-service and graduate teacher training, curriculum development and research work in mathematics education. Throughout her career, she has been deeply involved in the mathematics curriculum and teacher-training projects of the Philippine government. For the last four terms, she has been elected and re-elected Vice-President of the Mathematics Teachers Association of the Philippines.

KLAUS GALDA. Born: 1943. B.S. in mathematics, Montana State University, United States; studied mathematics and philosophy at the University of Hamburg, Federal Republic of Germany; Ph.D. in philosophy (logic and foundations of mathematics), Stanford University, United States. He has taught mathematics and logic at the Catholic University of Rio de Janeiro in Brazil, and English at the National Language Laboratory School in Osaka, Japan. Since 1977, he has worked for Stanford University on the Nicaraguan Radio Mathematics Project, and on a number of similar activities in several other countries. He has been a consultant for USAID, Unesco and World Bank projects in primary-school education in Latin America, Asia and Africa.

ANDREW HERRIOT. Senior lecturer at the National Teacher Training College, Maseru, Lesotho. Honours graduate, Heriot-Watt University, Edinburgh, Scotland. The Lesotho appointment was for five years under the Aid to Commonwealth Teaching Science (ACTS) scheme. After six years of teaching in schools, he first became interested in 1969 in teacher education and has held various posts in Scotland (Hamilton College of Education), Ghana (Advanced Teacher Training College) and Lesotho (NTTC). His main interests are in the field of instructional materials for both pre-service and in-service teachers of mathematics together with techniques for assessment in teacher education.

PEGGY A. HOUSE. Associate Professor of Mathematics Education, University of Minnesota, United States. M.S. and Ph.D. in mathematics and physics education, Kansas State University. Professor House directs the mathematics education programme for secondary-school teachers at the University of Minnesota. Her publications include two monographs, three chapters in NCTM yearbooks, and numerous journal articles. Her principal interests are in teacher education, problem-solving, teaching gifted students and the integration of science and mathematics teaching. She is currently vice-president of the Minnesota Council of Teachers in Mathematics and a member of the Board of Directors and chairwoman of the publications committee of the School Science and Mathematics Association.

DAVID C. JOHNSON. Shell Professor of Mathematics Education, Centre for Science and Mathematics Education (CSME), Chelsea College, University of London, United Kingdom, since autumn, 1978. B.A. in Physical Science, Colgate University (1958); Ph.D. in Mathematics Education, University of Minnesota (1965). He previously taught at the University of Minnesota, United States. He has taught at all levels—primary, secondary and tertiary—and has been involved in the training of primary- and secondary-school teachers for the past twenty years. He began his work with calculators and computers in school mathematics in the mid-1960s. He directed and co-authored the research and development project *Computer Assisted Mathematics Program,* CAMP, 1964–1970, which included the publication of a set of pupil texts and teacher's guides for the age-range 12 to 18. The materials were on the philosophy that pupils write their own computer programs to study mathematical topics and solve problems. His most recent work is a book, *Explore Mathematical Ideas with Your Micro Computer: A Book for Kids Aged 9–90* (in press). He was also the first editor of the *Journal for Research in Mathematics Education,* JRME (Volumes 1 to 4). He has published numerous research and 'practical' papers across a range of topics.

REGINALDO NAVES DE SOUZA LIMA. Associate Professor of logic and modern algebra, Federal University of Minas Gerais (UFMG), Brazil. Graduated from UFMG, major in mathematics; Masters in Science and Mathematics Teaching, University of Campinas (UNICAMP). He is currently developing experiments in mathematics teaching at the primary- and secondary-school levels, through the Science Teacher Training Centre of Minas Gerais (CECIMIG) in Belo Horizonte.

MICHAEL MITCHELMORE. Senior Research Fellow, School of Education, University of the West Indies, Jamaica. B.A. in Mathematics, University of Cambridge (1961); Postgraduate Certificate in Education, Bristol University (1962); Ph.D. in Mathematics Education, Ohio State University (1974). He taught mathematics for several years at high schools in Ghana, where he played a leading role in a mathematics curriculum development project whose materials are still used extensively in West Africa and the West Indies. He moved to Jamaica in 1973 after two years' graduate study in the United States. After a period working on examinations at the Ministry of Education in Kingston, he took up his present position in 1976. His main research interest is the development of spatial visualization among children in developing countries and its relation to achievement in mathematics.

ALAN OSBORNE. Professor of Mathematics Education, Ohio State University, Columbus, Ohio, United States. Undergraduate studies at Earlham College; M.A. in Mathematics and Ph.D. in Mathematics Education, University of Michigan. He taught school mathematics for seven years prior to obtaining his higher degrees. His primary research interests concern children's formation of measure concepts and estimation strategies. Recently he directed the PRISM project that provided some of the data used by the National Council of Teachers of Mathematics in formulating *The Agenda for Action,* a curricular blueprint for school mathematics during the 1980s.

FIDEL OTEIZA M. Lecturer and researcher in the Department of Mathematics and Computer Science, Faculty of Science, University of Santiago, Chile. In this capacity, he co-ordinates the postgraduate programme in mathematics teaching of the university. As an experienced teacher of mathematics, he has specialized in the development of curricula. His main fields of interest are adult education, the design, development and testing of learning systems and the evaluation of education programmes.

MICHELE PELLEREY. Director of the Educational Department, Salesian University, Rome. Born: Genoa (Italy), 1935. Doctorate in Mathematics; specialization, Educational Psychology. He conducted many research projects on the learning processes of mathematical concepts and directed the important RICME project on elementary mathematics education. He is now involved in two projects: the first is concerned with the learning of informatics concepts and procedures at pre-university level and in vocational training, the second with the learning of mathematics by disabled children.

THOMAS R. POST. Professor of Mathematics Education, University of Minnesota, Minneapolis, Minnesota. He has published in all of the major journals dealing with research and methodology in mathematics education in the United States, as well as having co-authored two books, one dealing with the mathematics laboratory, the other dealing with interdisciplinary studies at the elementary and junior high school level. His research interests include early number concept formation and the development of rational number concepts in children of school age. He is currently co-principal investigator of the NSF supported Rational Number Project. This project is concerned with generating hypotheses on the nature of the impact of manipulated materials on the learning of rational number concepts.

BRYN ROBERTS. Teacher education and curriculum development in Papua New Guinea. M.A. in Curriculum Development, University of Sussex, England. His chief professional interest, however, is in mathematics education in developing countries. After serving briefly as a teacher in his home country, Wales, he took an appointment in a secondary school in Zambia. He then moved into teacher education and served successively in Botswana and Swaziland.

PETER SANDERS. Senior lecturer in Mathematics Education, University of the South Pacific, Fiji. M.A. in mathematics, Oxford and London Universities. From 1959 to 1968 he lectured in mathematics at the University of the West Indies and then at the University of Nairobi. From 1968 to 1976 he worked on teacher training projects for Unesco, at the University of Zambia and later at the University of Botswana, Lesotho and Swaziland. He came to Fiji in 1976. He has been active in the mathematics associations in the countries where he has worked, serving for periods as the chairman in Kenya, the treasurer in Zambia and the secretary in Fiji.

JAMES SCHULTZ. Associate Professor of Mathematics, Ohio State University, United States. M.S. in Mathematics and Ph.D. in Mathematics Education, Ohio

State University. He taught school mathematics for five years after completing undergraduate work at the University of Wisconsin. He specializes in the mathematical preparation of elementary-school teachers and has been involved with teacher training programs at several American universities and in Indonesia and Costa Rica.

HILARY SHUARD. Deputy Principal, Homerton College, Cambridge, United Kingdom. After undergraduate and graduate work in mathematics in Oxford and Cambridge, she taught in schools before becoming head of the mathematics department of Homerton College, a college of education now within the University of Cambridge, and she continues to work in mathematical education. She has recently been a member of the Cockcroft Committee, a government committee of inquiry into the teaching of mathematics in primary and secondary schools in England. Her major interest is in primary-school mathematics; she is also interested in the teaching of calculus and analysis.

RANDALL J. SOUVINEY. Associate Co-ordinator for the Teacher Education Program, University of California, San Diego, United States. Ph.D. in Mathematics Education, Arizona State University (1977). He has enjoyed a varied career in education over the past two decades. After serving as an elementary and secondary teacher for eight years, he joined the faculty at UCSD as a supervisor of teacher education. Over the past ten years he has authored over forty articles in various education and research journals. He has served as a reviewer for several journals and publishers, including *Investigations in Mathematics, School Science and Mathematics,* Goodyear Publishing Company and Scott, Foresman and Company. During 1979–81, he was supported by Unesco to serve as Director of the Indigenous Mathematics Project in Papua New Guinea and edited an extensive series of working papers reporting the results of the cognitive, ethnographic and curriculum development research carried out under its auspices. He is currently engaged in pre-service teacher education and is completing work on his fourth book, entitled *Learning to Teach Mathematics,* to be published by Scott, Foresman and Company in 1983.

P. K. SRINIVASAN. Former Fulbright Exchange Teacher of Mathematics in the United States and former Senior Education Officer and Senior Lecturer in Mathematics in Nigeria. He has been involved in mathematics education activities for over three decades. He is the author of recreational and enrichment books in maths for children and has written numerous articles on mathematics education appearing in *Mathematics Teacher* (India) and *Mathematics Teacher* (United States). He is interested in instant improvisation techniques in the teaching of mathematics in schools and in teacher training institutions. He has conducted over fifty mathematics expositions in India and abroad. He was founder and honorary secretary of the Ramanujan Memorial Foundation, and is a full member of the International Congress on Mathematics Education.

MARIA DO CARMO VILA. Assistant Professor at the Federal University of Minas Gerais (UFMG), Brazil. Degree in mathematics, Faculty of Philosophy, Science and Literature, Guaxupé; Masters in Science and Mathematics Teaching, University of Campinas (UNICAMP). She teaches linear algebra and beginning post-secondary mathematics courses. She is currently developing experiments in mathematics teaching at the primary and secondary levels through the Science Teacher Training Centre of Minas Gerais, in Belo Horizonte.

GERHARD WALTHER. Professor for Mathematics Education, Institute for Mathematics Teacher Training, Kiel, Federal Republic of Germany. Pursued studies in pure mathematics. He has worked since 1971 at Dortmund University in the field of mathematics teacher education, where he took his doctor's degree and made his habilitation (on the reading of mathematical texts). Further research interests include the psychology of mathematics learning, the philosophy of mathematics teaching, and strategies for teacher training.