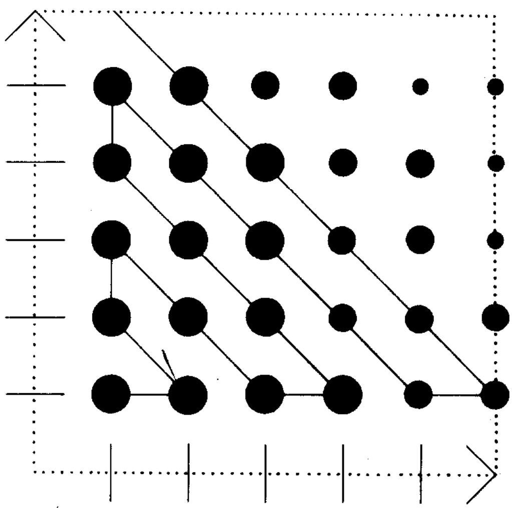
Volume 6

Studies in mathematics education

Out-of-school mathematics education

Edited by Robert Morris

The teaching of basic sciences Mathematics



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The teaching of basic sciences

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Preface

Mathematics teaching brings the image of blackboard and chalk, perhaps even calculators or microcomputers. This volume of *Studies in Mathematics Education* brings a different vision, away from the classroom. It examines various ways of encouraging pupils to learn mathematics in school through activities outside the classroom or school building, such as mathematics clubs, fair and competitions. The book also describes ways of bringing mathematics to those who are not, or perhaps have never been in school. Some of these ways are through popular mathematical articles in the press or on television, science museums, microcomputers and calculators in the home, all of which might be available for the general public who are interested in learning more mathematics.

It is hoped that this volume will be of interest, and above all useful, to those who wish to extend the learning of mathematics.

The next volumes will be devoted to the teaching of statistics and to the role of microcomputers in mathematics teaching.

Unesco wishes to express its appreciation to the series editor, Robert Morris, and to the many contributors to this, the sixth volume of *Studies in Mathematics Education*. The views expressed in the signed chapters are, of course, those of the authors and do not necessarily represent any position on the part of Unesco or of the editor.

Contents

Introduction

This the sixth volume of *Studies in Mathematics Education* makes a complete break with the classroom. It is devoted exclusively to mathematics education out of school. The subject is wide and it can be looked at under three broad headings: activities which are designed to encourage those who are still in primary and secondary education; the mathematics which is diffused by the media; and the various sources of mathematics other than the first two.

Out-of-school activities which are sometimes on offer to those who are in school include mathematics clubs, mathematics camps, mathematics competitions, mathematics in science or other fairs, and those visits and excursions which are specifically arranged to exhibit mathematics in human activity.

Mathematics emerges from the media sometimes deliberately, when, for example, regular instruction is given by radio or television, and sometimes by accident, when, for example, the leader-writer of a famous journal complains that 'one-half of all the children in school are below average in reading ability'. Then, of course, there are many examples of books and magazines which feature mathematical articles or, more often, mathematical puzzles.

The 'other sources' include museums of science with mathematical exhibits, mathematics courses which are orientated specifically to a particular occupation, mathematics education provided from a distant source, mathematics instruction in literacy classes for adults, and, of course, increasingly, the arrival in the home of calculators and computers which teach - even when they are treated as toys.

The three categories here briefly described account for the sub-division of what flows into its first three parts. The fourth part consists of a case study of out-of-school mathematics in Colombia.

Mathematics clubs are the subject of Chapter 1. Their essential purpose, as Saulo Rada-Aranda points out, is to supplement and enrich the regular fare of the classroom. In this chapter, the aims of such clubs are discussed, how they might be organized and what activities they might typically undertake. In Chapter 2, Barbara Rabijewska and Mieczysław Trad give an account of the summer camps in Poland which, for the last twelve years, have proved their worth as a means of nurturing the interests and abilities of young people of school age who are mathematically gifted. Their description of the camps gives a vivid account of their organization, the activities they provide and how these relate to the regular activities of the Pythagoras Club for Young Mathematicians to which the campers belong.

The next chapters are concerned with mathematical competitions, particularly those whose subsidiary purpose is to reveal representational talent which can carry the flag in an international 'olympiad'. In the first of these, Samuel Greitzer provides from his long experience an account of the growth and development of the Mathematical Olympiads of the United States and of the International Mathematics Olympiads. This account leads into an interesting discussion of the differences between 'talent' and 'flair' and of how countries can best cater for those in school and in the wider community who are mathematically gifted. His conclusion - that the size of a country is less important than its public esteem for mathematicians - is supported by the special case of Hungary. In Chapter 4, Professor Lê Hải Châu offers a clear and interesting account of how pupils in Vietnam who have a flair for mathematics are identified, encouraged and trained to compete in regional and national tests of mathematial ability. Professor Lê's account includes a set of problems actually used in a recent National Olympiad. Especially interesting is his discussion of the influence of the National Olympiads upon the teaching of mathematics in school generally.

The place of mathematics in the media is an elusive subject, and, in this volume, only one example is given - that of the accumulated experience of the work of the Open University of the United Kingdom. The discussion focuses upon the material broadcast in the context of the foundation course in mathematics which the Open University provides, and it includes the place of audio and video-cassettes. What is of special interest in the context of effectiveness is the finding that television broadcasts are useful, but that radio broadcasts are not. Or, to put the conclusion in a fairer light, that radio-transmitted material is no substitute for written material, though televised material can, in some instances, have the edge in this comparison.

This Open University mathematics course is followed by about 3,400 students and a feature of the account of it given in Chapter 5 is the wealth of examples of actual sequences used in the course to illustrate its method of teaching. So we are offered three examples of material used to prepare students for the course itself, while the way in which television broadcasts, audio-cassettes and written material are woven together is illustrated by the sequence entitled 'Functions and Numbers'. Finally, a sequence on carbon dating, leading to the definition of a^x for irrational values of x is quoted to convey the flavour of post programme material with which the course concludes.

The chapter which follows is on 'Distance Education in Mathematics'. Based upon a New Zealand experience, it makes a happy juxtaposition with the previous chapter in that it reinforces the findings of the Open University, namely that students who depend for their learning upon what comes through the mail almost invariably have to pass through three stages of intellectual growth: first, a process of becoming familiar with a unit of work in order to find out what it is concerned with - what the Open University calls 'exposure'; second, a stage of active effort to solve a sequence of problems, what the Open University calls 'experience'; and third, the stage of completion, which the Open University calls 'mastery'. This three-stage evolution of mathematical maturity has strongly influenced material design during the sixteen years since distance teaching of mathematics in New Zealand began.

Chapter 7 may be regarded as an interlude in that it is more specifically concerned with identifying talent than with catering for it in some specific out-of-school context. It draws upon the Hungarian experience, which, as has been said, is 'special', in its concern for out-of-school mathematics. Hence the relevance of Ferenc Genzwein's contribution.

Part III concludes with three chapters on mathematics provided particularly for those with fewer talents or those who missed their opportunities to learn mathematics: mathematics in literacy classes and mathematics training for work and *Family Math*. The former is based upon experience in Africa, Asia and the Caribbean, and it covers such issues as purpose, the educational and social value of numeracy classes, teaching methods, curricular content, linguistics considerations and the evaluation of adult progress. The conclusion is an interesting one: numeracy classes tend to be held in lower esteem than literacy classes, yet they tend to be more successful. Ideally, literacy and numeracy should go hand-in-hand and so re-inforce each other.

Mathematics Training for Work begins with a challenging question: what is 'work'? Upon the answer depends the difference between the mathematics used in work and mathematics conceived as a discipline. Training 'on the job' is then contrasted with training in an institutional context. Both have their own advantages and both their disadvantages. While the main concern of the chapter is for the employee at the operational level, the needs of the graduate entrant into industry are taken into account and this leads to a discussion of practice in France. To conclude, the chapter looks carefully at trends in industry and commerce and predicts that their mathematical needs will grow both in amount and in their level of difficulty.

Family Math provides skills and confidence to those groups of students who reduce their future options by prematurely dropping mathematics. Developed at the University of California's Lawrence Hall of Science as part of its programme to increase public understanding and enjoyment of mathematics and science, *Family Math* brings together parents and children to develop the mathematical skills needed for future education or work. While teaching parents to help their children with mathematics, a closer parent-school communication is fostered.

Part IV, as has been said, is devoted to a single case study, that of out-of-school mathematics in Colombia. This charts the growth over five years of a very determined and highly successful effort to influence the course of school mathematics towards a problem-solving orientation. What began in 1981 with a mathematics competition organized locally in the country's capital has grown rapidly into an enthusiastic acceptance of mathematics as a mental stimulant available to all members of the community, whether professional or artisan, urban dwellers or isolated farming folk. It is a particularly appropriate story with which to end this volume.

Part I

Activities arranged for the younger learner

1. Mathematics clubs

Saulo Rada-Aranda

Extra-curricular mathematical activities can have a number of different objectives, but their essential purpose is to supplement the mathematics education provided for children and young people in the formal system where the process of adapting school syllabuses to new trends in mathematics teaching is often slow. In this respect, out-of-school activities provide an excellent opportunity to introduce new ideas. Another purpose is to identify students with a special aptitude for mathematics. This can fill a void since, in many countries, there is no machinery for identifying outstandingly gifted children and even when they are discovered the school system does not offer effective encouragment and guidance. Again, extra-curricular activities can make mathematics more popular with students, with teachers and with the general public. Traditionally, a large proportion of the school and the adult population shows a degree of hostility towards mathematics. Children and young people should be offered the opportunity to participate in instructive and enjoyable activities, encouraging them to specialize in scientific and technical subjects, or, at least, ensuring that mathematics is not a disincentive when the time comes to choose a course of higher education.

Mathematics clubs offer a simple way of starting out-of-school mathematical activities which do not require a complicated organizational structure or involve high costs necessitating substantial external financing. Such clubs, established assocations of children and young people, require a recognized structure and suitably qualified adult advisers. Their aim is to organize activities contributing to the scientific and technological education of their members and of the community as a whole (SECAB-Unesco, 1985).

The objectives of a mathematics club

Mathematics clubs will formulate their own objectives, but these are likely to include the following:

To stimulate young peoples' interest in scientific and technical subjects by offering them the opportunity to develop their skills and talents in novel and entertaining ways.

To supplement students' formal mathematics training, introducing them to themes that are generally not covered in school syllabuses, and to help young people appreciate the importance of mathematics in society.

To stimulate creative capacity, a spirit of inquiry and a critical attitude in young people, developing values such as objectivity, intellectual honesty and the use of reflective and logical thinking when confronted with problems.

To promote activities and projects of common interest, encouraging a team spirit.

To provide the club's members with better mathematical training, enabling them to improve their performance at school and to be better prepared for higher education courses for which mathematics is required.

Membership requirements and organization

A mathematics club may be formed on the initiative of a teacher or of a group of students. But, as the aim is to arouse interest, membership should be voluntary. Moreover, the way in which the club is organized should ensure that all members take part in its activities on an equal footing. What is essential is that the young people who decide to join the club must realize that it involves new, well-organized activities that will take up time in addition to the time they spend on their school work, and that it calls for discipline in conforming to the agreed rules of the club.

If the club is started by a group of students, the members should be assisted and supervised by an adult adviser, who should be an enthusiastic, qualified teacher who has studied mathematics and who would therefore guarantee the seriousness of the club's work.

It is also important for the club to have the support of a school or of an educational institution that will provide a room where members can meet, a library and, if necessary, other assistance, such as the advice of those with experience in this kind of work.

The group may appoint a committee to draw up the statutes that will govern the club's activities. The statutes should be sufficiently relaxed and flexible so as not to detract from the club's vitality, but they should be clear about the requirements for club membership: age, academic level, institution attended, etc. They should also set out the composition of the governing council, the responsibilities of each member of it, the duration of their terms of office, the way in which the annual programme of activities should be planned and the possibility of allocating to members specific responsibilities for public relations, for the co-ordination of other extra-curricular mathematical activities such as exhibitions, competitions and mathematics olympiads, for publicizing the club's activities, etc. The statutes will also contain rules on the attendance of members at club meetings and their participation in club activities, on the annulment of periodical subscriptions should the need arise, and so on. The statutes could also distinguish between different categories of member (active, collaborators, honorary, etc.). This would give some status to mathematicians, professionals and other members of the community who might support the club's activities in one way or another.

The committee will submit its draft statutes to the other members who, under the adviser's guidance, will assemble to consider and approve the draft, incorporate any amendments they consider necessary and elect the club's governing council.

Mathematics clubs are normally intended for secondary-school students (approximately 13 to 18 years of age) and sometimes for university students. Nevertheless, the importance of encouraging such activities in primary schools and in teacher-training institutions has been recognized, and suggestions have been made for the organization of clubs and on activities suitable for those levels (Srinivasan, 1984).

Type of activities to be undertaken

If the organized activities are to attain the club's objectives, they should meet the following requirements:

They should be interesting, enjoyable and scientifically sound. This will ensure all members' active and willing participation with no obligation other than that voluntarily undertaken to observe the club's statutes.

The content of the activities should be educational, effectively improving members' mathematical knowledge.

They should be varied enough to cater for different levels of interest and maturity and different interests over and above the shared interest in mathematics.

The limitations on the time available to members should be taken into account, so that extra-curricular activity does not interfere with members' formal education or other essential aspects of student life.

The atmosphere should be different from that of the classroom. The relationship between the club's adviser and the members is different from the traditional teacher-pupil relationship, since, in the club, it is the active members who should take the initiative and play the most active part in the planning and organization of activities.

Some typical activities

The study of particular aspects of mathematics. One of the essential functions of a mathematics club is to study topics that are not normally covered in the basic school syllabus. These topics may be investigated (with the adviser's guidance) individually or in groups, and they will be adapted to the members' age and mathematical knowledge. The investigation may take the form of activities such as problem-solving, the manipulation of apparatus or applications to other sciences or to everyday situations. Themes need not be studied in exhaustive detail. Members may subsequently explain their work to the rest of the group.

The themes selected may include some that are relevant to a very large number of problems but which are treated very superficially in traditional syllabuses. Two such possibilities are geometry and the theory of numbers. Other themes may be chosen for their usefulness to the student at a later stage. These might include the calculus and the study of some basic aspects of current mathematics such as functions and relations, stuctures, etc. In the same category are those aspects of mathematics that have more practical applications such as probability and statistics, the theory of graphs, linear programming, matrix algebra and financial mathematics. An excellent list, with samples of applications to mathematics teaching and learning in secondary schools, is contained in the report of a Unesco-sponsored meeting held in Montevideo (Unesco, 1974).

The programming of lectures, workshops and film shows. Such activities are important not only to meet members' expectations and sustain their interest, but also because they make it possible to extend the club's activities into the wider community. Lectures and workshops may be planned individually or in cycles. They should bring into the club mathematicians and other professionals who use mahtematics in their work. The subjects chosen should be of interest to most of the participants; they might include topics connected with the history of mathematics, the life and work of prominent mathematicians, problems of recreational mathematics, the relationship between mathematics and art, the importance of mathematics in social development, the mathematical requirements of society in the future, and mathematics and music. Topics connected with mathematical method and with science may also be discussed. These will illustrate the type of reasoning employed in mathematics and in the natural sciences, mathematical logic and problem-solving.

Films, video recordings, slides and other audio-visual material can also help to promote interest in mathematics in the community as a whole. Many institutions already possess such resources. Club members may approach educational institutions, educational resource centres, departments of educational technology, embassies and any other institution that may be willing to lend. Film shows to which members of the public are invited should be planned well in advance, and time should be allowed for additional explanations, as well as questions and answers, if a wider community interest is to be attained.

Study groups to prepare other out-of-school mathematical activities. Clubs are the simplest to organize of all the possible types of extra-curricular activity. They can consequently be used as the starting-point for the organization of school open-days, of mathematics Olympiads and other activities.

The club could promote a mathematics open-day at which its members, with the adviser's help and guidance, can display and discuss the work they have done. The club can also sponsor mathematics Olympiads, either for the school to which it is attached or for students from other institutions with (or without) mathematics clubs. In this context, the adviser may organize training courses and workshops on ways to solve mathematical problems with a view to preparing members who are keen to take part in these competition. The levels of difficulty of the problems set would naturally reflect the students' ages and attainments. Such courses can also be of tremendous help to students with their school work and can encourage them to acquire the skills and talents they need to face new situations both in mathematics and in life.

Organization of a library. It is important that the mathematics club should have a small library of books and magazines, as distinct from the usual range of school textbooks. From its inception, the club should acquire a few publications, as gifts probably, of the parent school or institution. One of the members of the club may be put in charge of running the library and of making additions to the original stock, whether through members' contributions, by purchases from the club's funds or as assistance requested from external sources.

Publication of a review or bulletin. The mathematics club should not only publicize its activities within the school community in which it operates, but should also circulate information to other professionals and institutions that may be able to assist in making its programme a success.

Reports of club activities should be prepared and visits to the club arranged. Above all, a news sheet, in the form of a magazine or bulletin, should be issued at regular intervals. Initially it could be quite short, but at the same time sufficient to keep individuals and groups concerned informed of the club's activities. The same bulletin can be used to exchange news with other clubs, both at home and abroad. One club member should be made responsible for bringing out the bulletin and given guidance in doing so as necessary.

Maths workshop. At a later stage, club members might consider providing themselves with a mathematics workshop or laboratory. Such a facility would enable them to produce material illustrating in a practical way certain concepts and mathematical properties, extremely useful in certain subjects, i.e. geometry. Such material could also be used in the assembly of projects to be presented at open-days, in competitions and in other extra-curricular mathematical activities.

The workshop could also become the souce of teaching aids for use in the parent school and in other educational institutions.

Evaluation

In accordance with the club's objectives, it is important to assess to what extent its activities improve its members' attitude to mathematics and enable them to reap more benefit from their formal studies. It would also be helpful to establish personal files for all members. These could carry information about the members' performance prior to joining the club and any special traits or abilities they develop as they go along and which could be evaluated at a later stage.

As with other out-of-school mathematical activities, a check should be kept on the extent to which students have been able to take part in a project, have exhibited creativity and have developed their critical faculties. It should also be determined whether the students have learned to work as a team with teachers and other club members and whether such fundamental qualities as team spirit, discipline, responsibility and tenacity have been developed or reinforced. These are some of the points to which the counsellor should pay special attention when evaluating the development of the attitudes desirable in young participants in the course of their work. The club's programme of activities may be evaluated in terms of the work accomplished by its members and of their participation in open-days, mathematics olympiads and other extra-curricular activities.

Such an assessment should be made public. In addition to helping to consolidate the club's activities, it provides feedback which can be taken into account when drawing up future programmes of work. It might be circulated in the bulletin and at mathematics club conventions, where experiences can be exchanged with other like-minded groups.

References

SECAB-UNESCO. 1985. *Manual para el fomento de las actividades científicas y tecnológicas juveniles*. Bogotá, Permanent Executive Secretariat of the 'Andrés Bello' Convention (SECAB).

SRINIVASAN, P. K. 1984. Cultivation of Climate through Mathematics Clubs. In: R. Morris (ed.), *Studies in Mathematics Education*. Vol. 3: *The Mathematical Education of Primary-school Teachers*, pp. 229-34. Paris, Unesco.

UNESCO. 1974. *Las aplicaciones en la enseñanza y el aprendizaje de la matemática en la escuela secundaria*. Montevideo, Unesco Regional Office for Science and Technology for Latin America and the Caribbean.

2. Mathematical camps

Barbara Rabijewska and Mieczysław Trad

A fundamental and necessary skill for effective teaching and learning appears to be the ability of the teacher to give and to receive appropriate responses to questions put and asked. These stimulate the growth of the knowledge of the brightest pupils. But such growth is not only determined by teaching methods; it can also be maintained by specific techniques of interaction, by the proper organization of the didactic process or, in other words, by reinforcing the most relevant teaching strategies.

The needs and interests of pupils, which motivate and stimulate their efforts, can change as they develop and, when perceptively guided, they can stimulate the pupils' activity and can foster the will to undertake and perform tasks which might otherwise prove fairly difficult for them to accomplish. These are the motivational factors favouring the desired growth of pupils' abilities from innate to well-developed. To secure for the brightest pupils propitious developmental conditions, it is necessary, as a rule, to take special care, not only in the normal classroom, but during out-of-school activities as well. Extra-curricular activities assume various forms. Advancement can, for example, be made by team work in mathematics clubs or in circles of mathematical interest. And such advancement can be further perfected in mathematical camps or in summer schools of mathematics.

In the United States, apart from extra-curricular activities, specific techniques are introduced into the classroom. One of these, for instance, is the individual approach to bright pupils. This involves the careful selection of learning material and adopting teaching methods appropriate to the individual needs and abilities of the most gifted pupils. In addition, after school, special classes are provided for such pupils at school clubs or at universities. When holidays come, special classes are held in mathematics summer schools or in mathematical camps for the selected group of brightest pupils (Hoyle and Wilks, 1974).

In the Soviet Union, in turn, there are secondary level boarding schools sponsored by the universities of Moscow, Kiev, Leningrad and Novosibirsk.

These schools are for pupils who have been found to be particularly gifted in science. Mathematics summer schools, which began in 1963, help to identify suitable pupils for the boarding schools (Volkov and Rubinov, 1970) and (Vavilov and Zemiakov, 1979). Mathematical camps are also organized for children from rural communities (Gorbunova et al., 1975). Both camps and summer schools aim to develop interest in mathematics, to build up habits of self-confident work, to further the ability to solve unconventional problems, to broaden the scope of knowledge and to develop an aesthetic appreciation of mathematics.

Mathematical camps are held in Czechoslovakia and in the German Democratic Republic as well. In the Kosice district of Czechoslovakia, for instance, camp participants are chosen from among those pupils who do best in solving problems which are sent to them each month, six at a time. Mathematical camps organized in the German Democratic Republic serve to prepare participants for mathematical olympiads.

In south-western Poland, several mathematical camps for pupils aged 13 to 14 have been held. The idea behind these camps, which began in 1978, was to enhance and to deepen the participants' interest in mathematics. In the meantime, other mathematical camps have been held.

In 1974, the so-called 'Pythagoras' Club was established in the Zielona Gora district of Poland for young mathematicians. Its objectives (which are equally relevant to mathematical camps) are as follows:

To develop members' interests and abilities.
To foster creative and original thinking.
To enhance self-confidence in thought and in work.
To work out an appropriate system of selecting mathematically gifted pupils.
To broaden the scope of members' knowledge.
To prepare for further education in secondary-level schools orientated towards mathematics and physics, in university-level studies in mathematics and in economics, and in polytechnic faculties of universities.
To shape socially-acceptable behaviour.

Club members engage in a variety of activities. The following are some of them:

Taking classes in teams, such as paths leading to mathematics, brain-teasers, semi-seminars, and mathematical tournaments;.
Tasks assigned to corresponding members for each school year include. working up assigned topics, writing short treatises, solving problems, and entering tournaments and competitions.
Holding sessions to which scientists from universities and research institutes are invited.
Undertaking individual work for which a medal of the Pythagoras Club can be won.
Reading popularized scientific publications and mathematical journals.
Going on mathematical trips and expedition.
Attending cultural and entertaining events.

Members are enrolled in the club according to their achievement in schoolwork, the recommendation of their mathematics teachers and the approval by the headmaster of a pupil's application. Membership is restricted to primary and secondary school pupils, aged 12 to 18. Some work in teams on the premises and some are corresponding members.

Friendly relationships have been established between the Pythagoras Club and a club from the Cottbus region of the German Democratic Republic. This permits the exchange of ideas, views and experiences of working with mathematically gifted pupils. There is also a mutual exchange of members when olympiads, competitions or mathematical camps are organized.

It is necessary to point out that mathematical camps have an important bearing upon the day-to-day activities of the club. The work of the club is continued on mathematical camps. Conversely, the activities of the camp have implications for the activities of the club in the following school year.

A mathematical camp for club members was organized for the first time in June 1975. Each year, several members from particular teams and from a specified age group take part in the camp. Priorities are given to the best pupils from a given team. The camps are sponsored by the local education authorities. The latter bear the expenses of the camps. Participants in a camp will contribute between 6 and 50 per cent of the camp expenses, as a function of family income. A camp lasts for about seventeen days. As the accommodation and the social facilities are provided in the same place, the organization of the didactic and other activities is a simple matter; only the reading room and the reference library have to be set up at the start of the camp.

The actual numbers going to camp is shown in Table 1.

Table 1. Numbers of club members attending mathematics camps

Age	1975	1976	1978	1979	1980	1980	1981	1982	1983	1984	1985
13-15	53	78	64	-	30	44	56	68	-	65[1]	60[2]
16-18	-	-	-	65	-	31	18	-	57	-	-

1. Includes eleven pupils from German Democratic Republic.
2. Excludes twelve members who attended a camp in the German Democratic Republic.

Before a camp begins, all matters to do with its organization have to be settled. A framework for the programme is devised. Camp participants are divided into groups and subdivided into teams according to their interests and abilities.

For several years the camp participants used to be divided into three groups, Alpha, Beta and Gamma. These were in turn each subdivided into three teams. The nine teams were named in such a way that their initial letters could form the name Pythagoras, or *Pitagoras* in Polish spelling, as follows:

P entagram
I deal
T orus
A xiom
G roupoid
O perator
R adian
A bacus
S implex

Three years ago some changes were made. The camp participants are now divided into four groups, Alpha, Beta, Gamma and Delta; these are each subdivided into two teams and the eight teams are given the names of curves: Asteroid, Cycloid, Cardioid, Lemniscate, Nephroid, Rosette, Strophoid and Tractrix. The division into small teams of seven to eight pupils was found to be beneficial; participants became more active and independent in their mathematics work. Teams choose their leaders, called captains, who make up the camp council.

The activities at the mathematical camp for bright pupils fall into three categories: mathematics, culture and entertainment and sport, tourism and recreation. Each category is considered to be equally important. The nine-hour daily schedule is broken up into four sessions: 9 a.m. to 1 p.m.; 2.30 to 3.30 pm; 4.30 to 6.30 p.m.; and 8 to 9.30 pm. Of these nine hours, about four are devoted to mathematics, one and a half to culture and entertainment and three-and-a-half to sport, tourism and recreation. The first session of the day is devoted almost entirely to work on mathematics.

Certain methods and forms of activity have been worked out and improved upon in the light of experience, and we now have a list of twenty-five activities which we have found to be the most effective for bright pupils. (Rabijewska and Trad, 1985*a*.). The twenty-five are briefly described in the next section, but it will be realized that the collection is constantly being enriched.

Details of camp activities

Meeting scientists from universities and research institutes. Meetings between scientists from universities and camp participants assume the form of a lecture or a talk on some interesting mathematical problems. These are, of course, adapted to the level of participants' knowledge. The lecturer's aim is to develop pupils' interest in mathematics and to display its intellectual attractions. The personality of a speaker is of great importance, as are the style of presentation and the choice of particuliar problems. The idea is to stimulate further interest in the problems being discussed.

Mathematical evenings. Mathematical evenings are designed to present mathematics as a medium of entertainment. Such an evening is usually divided into two parts. The first part may include stories, curious pieces of information, discussions, an interesting lecture or meeting a well-known mathematician. The second part is usually devoted to mathematical tricks, games with numbers, sophisms, examples of quick calculations, poems, anecdotes or stage productions involving mathematics. During mathematical evenings, youngsters become more acquainted with various branches of mathematics in an informal way.

Mathematical quiz. A mathematical quiz is a competition in the presence of an audience. Twelve to sixteen pupils do the quiz; the rest observe. Those doing the quiz solve a prepared set of problems. The quiz can be divided into two or three stages to enable a greater number of persons to take an active part. Moreover, by using three or four blackboards, the same questions can be tackled by several persons simultaneously. After the quiz, a jury awards points to those who have taken part.

Brain-teasers. Brain-teasers is the name used for certain types of classes which may be conducted in groups or with the classes as a whole. These classes are meant to complement, by further explanation, the problems discussed at the meetings to which scientists from universities or research institutes are invited. They may also be separate classes devoted to a given problem.

Semi-seminars. In semi-seminars, pupils take the floor and give an account of some topic that has interested them. The purpose is to encourage the pupils to read popularized scientific books, magazines or journals on mathematics and to accustom them to independent work on mathematical texts or to induce them to gather books for their own mathematical libraries. For the person presenting his report, the only criterion for the choice of the subject-matter is its ability to arouse the listeners' interest. The topics presented by the speakers, often curious pieces of information, usually prove very interesting indeed. Before the presentation takes place, the pupil posts on the notice board his name, the problem to be dealt with at the semi-seminar and a bibliography.

Brainstorming. Brainstorming is a form of activity which seeks to stimulate an awareness of certain mathematical problems and of the ways they can be developed. The methodology of these classes involves creating situations in which the pupils are taught to devise and to formulate a new problem using one which they already understand and have solved. The idea is to develop in a pupil the ability to formulate problems at a constantly increasing level of difficulty. In this way, it is hoped that his mathematical activity will come closer to that of a creative mathematician. In short, brainstorming is meant to develop structuring abilities in the creative work of a pupil.

Guiding texts. When used in mathematical camps, guiding texts enable pupils to solve a difficult problem or to prove a theorem with the help of key elements of a solution or a proof. These key elements are written down on separate sheets of paper and numbered; the pupil provides the full solution of the problem or the full proof of the theorem, by arranging the key elements in their proper order. One set of sheets may contain the key elements of several problems to be solved or theorems to be proved, within the same thematic range or where the same mathematical symbols are used. Problems for guiding texts should be carefully chosen so as to include, principally, difficult or crucial problems, the solutions of which become possible by making use of guiding texts.

Olympian Club. The Olympian Club is an activity aimed at preparing pupils for participation in competitions, tournaments or mathematical olympiads. At meetings of the club, various ways of solving a problem are thoroughly discussed, special attention being paid to problems of methodological nature. The pupils are then expected to acquire some fundamental competence, and to know and to put into practice those typical steps of procedure which seem to be indispensable in whatever work they may have to do.

Mathematical games. Such games have a mathematical content and the rules are based on out-of-the-ordinary mathematical structures. The winning strategy may then depend upon discovering the property of such a structure or upon solving mathematical problems; it may also depend upon proper use being made of formerly-acquired knowledge. The activity of the pupils is seen in the ways they search for the successful strategy. Games are conducive to fostering both intellectual activity and interest in theory. The will to win often motivates the participants.

Consultations at duty hours. During duty hours, teachers are on hand to clear up any personal difficulty a pupil may have. A pupil may, for instance, have failed to follow the argument given for a particular result in a mathematical competition. On these occasions, a pupil can ask any question and discuss any problem he is interested in or has trouble in dealing with. The consultations are meant to guide the pupil in an individual way; they also serve to stimulate and develop the pupils' interests and abilities.

Adventure with mathematics. Adventure with mathematics is an event which usually takes place in the open air and is, as a rule, of a competitive character. Particular teams of pupils compete with one another to score as many points as possible. These are characteristically awarded for success in solving problems with the help of such objects as squares, cubes, sticks, dominoes, chess men or similar concrete material.

Marching, called 'Sports and Mathematics'. Patrol marching under the motto 'Sports and Mathematics' combines recreation with mathematical activity. While marching, particular teams perform proficiency tasks on route and solve various mathematical problems as well.

Stalking game. The stalking game is organized within a clearly delimited area of woodland. Members of the hiding team go to the forest first and mask their hiding place as well as possible. Members from the seeking team set out later in search of the first team. If someone is found, the finder sets him or her a problem. If the problem is solved, he or she loses a 'small' point. But if he or she fails to solve the problem, the loss is a 'big' point.

The winning team is the one whose members score the most 'big' points. It often happens, however, that the 'small' points are the decisive ones. The seeking team has to devise the problems to be solved by the hiding team.

Maths-ringo. Maths-ringo is an activity demanding certain physical skill. The pupils toss ringoes (rings) at concentrally-placed stakes. At the same time, they are busy with solving interesting mathematical problems.

Mathematical competition in teams. A team of pupils solves problems from the set prepared before the competition began. The problems can be solved together as a team, in smaller groups, or by individual members. At the end, the final solutions are formulated and presented by the team.

Mathematical league. Particular teams devise and formulate problems one after another and submit them to the opposite team to be solved in a fixed time.

Mathematical match. At a certain time, but not later than an hour before the match, the teams are given a set of problems which are to be solved by them during the match. About 75 per cent of problems are likely to be solved then. The match is usually played with an audience and three referees award points for solution publically.

Circle and Cross. This game, which is also called 'noughts and crosses', is best suited for two pupils or two teams to play. In either case, the game may be played in public or without an audience. For this game, a set of problems is divided into nine thematic groups, e.g. construction problems, vectors, theory of combinations, functions, geometric problems with proof, logical puzzles, numerical inequalities, dissection of geometric figures and divisibiity of numbers. The rules are the same as those in 'noughts and crosses'. A pupil or a team asks for a problem from one of the nine thematic groups. If they solve the problem, a circle or a cross is marked in the space which represents the particular thematic group of problems. The game ends with a win by the pupil or the team making a line with three circles or three crosses marked vertically, horizontally or diagonally.

Mathematical problems in foreign languages. Individual pupils, groups or whole teams are given problems or other mathematical text in a foreign language. This translate into their native tongue and, if there are problems, they must solve them.

Competition in pairs. Particular problems are solved by two pupils, but they must present one agreed solution.

Individual competition. Mathematical competitions and olympiads are tests of individual ability in which the participant is obliged to solve unaided a given set of problems under the control of a jury. Individual competitions then provide useful experience for those who hope to be chosen to take part in the big event.

Masters competition. Those who do best in an individual competition become 'masters' and subsequently compete with one another in the 'masters competition'.

Problem of the day. The 'problem of the day' is placed on the information board and is meant to be solved during the day. Pupils who solve the problem take part in a lottery and the winning ticket receives a book as a reward.

Formulating problems. Pupils compete with each other in devising and formulating problems. Problems that have been properly formulated are given publicity on the information board.

Mathematical information board. The information board not only provides camp participants with information; it also carries a certain amount of mathematical content, which is designed to teach.

The above set of twenty-five forms of activity can be classified according to various criteria and analysed from different points of view. The presentation is not illustrated with examples of relevant mathematical activity since such examples are to be found in a separate study (Rabijewska and Trad, 1985*b*.). The list of suggested forms of activity which are suitable for work with bright pupils is evidently far from complete. A constant search must go on for new and more attractive activities to be included in the camp curriculum since their sole purpose is to enhance pupils' interests and abilities. Innovations, then, are made year by year; in what follows, we give examples of new forms introduced into the camp activities in 1984 and 1985.

Mathematical hockey. The objects which characterize this form of camp activity are cardboard pucks similar to those used in ice hockey. The pucks have numbers and they are held by the chief referee. Non-standard mathematical problems are written on the cardboard pucks. Two teams of six pupils (three forwards, two backs and the goalkeeper) play. Each team can have reserves who may be substituted during the match. The team captain serves the puck and tells the opposite team its number. The puck passes successively to the forwards, the backs and the goalkeeper. Each player is allowed to keep the puck for 30 seconds only. During that time, the team is expected to solve the problem. The solution is assessed by two referees. If none of the players solves the problem, the opposing team scores 1 point. If they do solve the problem, the opposite team fails to score. Next, the opposing team's captain serves the puck; the procedure is repeated until the end of the match. A whistle interrupts the match or signals that the puck should be turned over to the other team. In the case of unfair behaviour, prompting or disturbing a player or a team supporter, the referee can exclude the offending player for 2 or 5 minutes. Note is made of any particular player who is particularly outstanding as a problem solver.

Outdoor mathematics. These are practical tasks which essentially involve working out of doors. The pupils can for instance be asked to measure the distance they have covered and so determine their mean speed of walking. They can calculate the width of a river or the height of a tree. They can assess the distance to some remote point or calculate the approximate number of trees in the forest. Another task may be to draw a map or make a plan.

Maths-sports. The principles are similar to those in maths-ringo, but, instead of tossing rings, the pupils use balls, darts or bowls to shoot at particular goals.

Reports on individual work. Pupils are asked to prepare written reports on their individual work in mathematics during the school year or in a camp.

Work in a reading-room. In order to be well-prepared for classes, pupils are advised to work on their own in the reading room. The tutor or designated pupils tell camp participants what is available in the reading room, including reports and diaries of all preceding camps. In this way, pupils gain experience in extracting information from books or other reference material.

Since the camp participants are also members of the young mathematicians' club, Pythagoras, all camp activity is meant to be an extension of the work of the club. A careful choice of activity will greatly enhance the interests and abilities of the pupils. Attractive and stimulating activities will motivate and enliven camp participants. Such forms of activity as semi-seminars, consultations at duty hours, reports on individual work, work in the reading-room, individual competitions, participation in quizzes and formulating problems build up self-confidence, while the study of mathematical books and journals makes for self-reliance. Team competitions, mathematical matches, mathematical hockey, competitions in twos, and noughts and crosses teach camp participants how to co-operate or to compete with one another. They gain a sense of self-confidence and satisfaction if they succeed. Meeting scientists from universities and research institutes and mathematical evenings make camp participants better acquainted with mathematics. Other activities, such as adventure with mathematics, sports and mathematics and mathematical games are meant both to teach and to entertain. But, of course, all these activities are subordinated to the main intention, that of developing and reinforcing mathematical interests and abilities and shaping socially accepted behaviour.

The camp is over, and the participants go back to their everyday school activities, making use of their camp experiences and acquired knowledge in the normal classroom. The methodology and forms of work devised and verified in a camp add considerably to the didactic process in the school year to follow.

The camp tutors are students from the institutes of mathematics who are being trained to become teachers after graduation. While working with bright pupils on mathematical camps, they acquire indispensable teaching experience likely to prove very useful in their future work. In addition, meeting scientists from universities and research institutes is profitable for both the camp participants and the tutors.

References

HOYLE, E.; WILKS, J. 1974. *Gifted Children and Their Education*. London, Department of Education and Science. (Education Information.)

GORBUNOVA, V.A.; LOMAKIN, Ju.B.; SAVELOVA, G.E. 1975. Letnije matematiceskije skoly dlja ucascihsja sel skih skol [Mathematics Summer School for Pupils from Rural Communities]. *Matematika v skole* [Mathematics in School], No. 3, p. 68.

RABIJEWSKA, B.; TRAD, M. 1985*a*. Forms of Work on a Mathematical Camp and a Fragment of Methodology of their Study. *International Journal of Mathematical Education in Science and Technology* (London), Vol. 16, No. 1, pp. 73-81.

——· 1985*b*. A Mathematical Camp for Bright Pupils. *Educational Studies in Mathematics* (Dordrecht/Boston, Mass.), Vol. 16, No. 1, pp. 41-57.

VAVILOV, V.V.; ZEMIAKOV, A.N. 1979. Iz opyta raboty letnei fizyko-matematiceskoi skoly pri MGU [Some Examples of the Work of the Mordvinian State University Mathematics and Physics Summer School]. *Matematika v skole* [Mathematics in School], No. 4, p. 61.

VOLKOV, V.A.; RUBINOV, M.M. 1970. Pervaja letnjaja skola na severo-zapade SSSR [The First Summer School in the North-West of USSR]. *Matematika v skole* [Mathematics in School], No. 6, p. 91.

3. Mathematical contests and olympiads

Samuel L. Greitzer

For some youngsters, athletic activities provide an outlet for excess energy; for others, it may be music or, perhaps, chess. There are, however, a surprisingly large number of youngsters for whom solving mathematical problems plays this role. Sources of such problems can be newspapers and periodicals. In addition, there are mathematics contests of various levels of difficulty. One example is the Eötvös Contest, held in Hungary since 1894. Others are the so-called 'Scholarship examinations' held in the United Kingdom and the *Concours* in France. These were respectively used to determine entrance to Cambridge University and to the *Grandes Ecoles*. Here, we shall restrict our discussion to contests that have more general purposes.

We do not have to explain why mathematically able people are needed for scientific investigation, engineering, business, etc. We will accordingly proceed to describe some forms of contests and then try to explain why they have been so successful.

One of the largest and most successful mathematics contest would seem to be the American High School Mathematics Examination (AHSME). This grew in 1959 from being just a local contest held in New York City into a national one. Now about 400,000 students compete in it annually. Moreover, there are other countries that use the same contest. Colombia uses a Spanish version of the contest. In this sense, this contest has actually become an international one. Other countries offer similar tests. The British use the British National Mathematics Contest; Poland has its own Mathematical Olympiad of Poland; and Sweden has its Swedish Mathematics Competition.

The International Mathematical Olympiad

Then, of course, there is the International Mathematical Olympiad (IMO). This began in 1959, when Romania invited other eastern European nations to take part in a jointly sponsored test of Mathematical talent. At first, seven countries responded to the invitation: Bulgaria, Czechoslovakia, the German Democratic Republic, Hungary, Poland, Romania and USSR which sent a token group of students. Participation was restricted to secondary school students of not more than 20 years of age. The IMO grew. Finland entered in 1965, France, the United Kingdom, Italy and Sweden in 1967, and the United States in 1974. By 1985, thirty-eight nations were taking part. The problems are based on the content of secondary school mathematics, but they require ingenuity to solve. The IMO is spread over days, with three problems presented each day. A time limit of four and one-half hours is set each day to solving the problems.

Each participating country is asked to submit problems for possible inclusion in the IMO. From these, a definitive set of six problems is chosen by representatives organizing the olympiad. When the problems have been tackled by the contestants, the delegates from each country grade the work of its own competitors. This grading is then checked by a group of co-ordinators from the host country. Prizes in the form of special certificates are awarded, and particularly ingenious solutions may receive special mention and awards.

Each year, a different country acts as host and invites the others to attend. All respondants are guests of the host country and have the opportunity for sight-seeing when time is available. The atmosphere is relaxed at first. Excitement builds up as results are displayed; it ends with a gathering when students and delegates relax in friendly fashion.

National contests

We now consider the content and purpose of some contests, beginning with the AHSME. This has as its main purpose the promotion of interest in mathematics on the part of students. It serves also as a means of checking the progress of participants and of promoting the art of teaching. The examination usually comprises thirty short-answer questions, starting with a very easy one and ending with a few more difficult ones. Students are free to enter the contest regardless of their grade in school.

Since 1972, this contest has also been used in the United States Mathematical Olympiad (USAMO). More recently, an examination called the American Intermediate Mathematics Examination (AIME) has been constructed to improve selection for the USAMO. Thus, the AIME and the USAMO were constructed to select team members for the IMO. Following the USAMO, it has become customary to arrange a 'training session' of three weeks for students who are or may be members of the IMO team. It includes instruction in additional mathematics as well as training in problem solving. A number of countries have followed the United States in instituting similar training sessions. It is surprising what can be accomplished in only three weeks of concentrated work.

The Mathematical Olympiad of Poland is intended to stimulate interest in mathematics and to induce superior students to enter universities and polytechnic institutes. This is not like the American contest. It takes place on three levels. At level one, problems are sent to about 1,500 students.

These they solve and mail to a central point at the rate of one problem per month. Levels two and three involve about 350 and 70 students. Their task is to solve three problems on each of two days. Winners may be selected for the IMO and will also receive special academic benefits.

Similarly, in the United Kingdom, there is a second round, in addition to the contest already mentioned, called the Further International Selection Test (FIST). This test determines the British team for the IMO. The Netherlands also sets a second round of problems to about 100 invited students.

It would seem that there are about as many countries holding just one contest as there are with two or three rounds. The objectives, too, would seem to be much the same. These are, first, to identify mathematical talent and to encourage the students who have it. Second, to help to select a team for the IMO. A possible third aim is to influence school curricula so as to induce changes which would make for the better preparation of students. As an example, there was one problem proposed recently that involved the concept of homogeneity. One country objected to this problem because this particular concept was not taught in their schools. The problem was accordingly withdrawn. Now such a problem would be acceptable if it satisfied all other conditions for a good problem. It may be assumed that homogeneity now appears in the curriculum of that country.

The talented and the gifted

At this point, we digress to discuss the difference between possessing talent and being gifted. As we see it, a talented student is one who has an excellent memory and who, having once seen a problem, can usually apply what he has seen before to the solution of a similar problem. By a gifted student, however, me mean one who, seeing a problem, is able to apply both what he knows and his imagination to solve a problem in what is often an unexpected and ingenious way. An example will illustrate the difference:

Solve, in natural numbers, $a^3 - b^3 - c^3 = 3abc$
$a^2 = 2(b + c)$

The talented student who has seen this sort of problem before is likely to elimate a and then work with the Diophantine equation which results.

The gifted student, on the other hand, might well argue as follows:

1. a^2 is even, so a is even, and a can be 2, 4, 6, 8, and so on;
2. $a > b$ and $a > c$ (from the first equation);
3. so $2a > b + c$, and $4a > 2(b + c)$;
4. hence $4a > a^2$ and $4 > a$;
5. therefore $a = 2$, and, finally, $b = c = 1$.

By taking this route he solved the problem mentally. We shall refer again to this difference between having talent and being gifted.

It should be obvious that devising suitable problems is no easy matter. In preliminary contests like the AHSME, one objective is to identify what we call 'manipulative ability'. The first few exercises (disguised as 'problems') test the student's ability to perform simple algebraic and other tasks. The problems become more complicated in the middle of the test and the last few do require special talent or a gift for mathematics. We find a similar gradation in the contests of most countries. It follows that suitable problems require skill to invent. They must be based on the secondary school curriculum, but they cannot be merely additional problems. Nor can they be of a routine nature. Indeed, the constructors must be as ingenious as the students they hope to discover. In some countries, members of the mathematics faculty work together to prepare problems at the request of the authorities responsible for the contest. In other countries, the problem committee may consist of interested mathematicians. A committee may even include a former contestant who has continued his or her work into the university.

How does one recognize mathematical talent or genius? It is rare to detect such gifted students as, say, Gauss, whose ability manifested itself when he was only 3. In most cases, the signs are far from obvious. However, the alert parents or teacher may find a child who can do work beyond the expected norm, who enjoys mathematics, likes puzzles, and is willing to spend a long time over problems in which he or she is interested. Most of us have had students who have discovered results which, while not original, were novel to the student. Good students may ask for further work in an area of mathematics which interest them. This is notable in the case of number theory, for example. Finally, such students exhibit superior work; they give better explanations and they show special interest in mathematics in general.

The teacher must be receptive to such evidences. Often a student will solve a problem in an original way and not 'by the book'. Such students should be watched and encouraged.

Catering for the gifted

When talented or gifted students are identified, the teacher's task is to develop their abilities. One way is to provide additional work for them to do. In some schools, there are special classes for the gifted. In such classes, the methods used are either to set more intensified work in addition to the normal curriculum, or to provide more rapid advancement. Sometimes, there are special schools that better students can attend. In the USSR, for example, students enter such schools as early as 8 years of age. When it has been determined that a student at least has talent, that student is placed in a special school. He or she receives the equivalent of a scholarship, and the family usually receives some sort of stipend. These are maintained as long as the student continues to show ability, but are removed if he or she does not do so.

We visited one such school where students were spending the summer doing mathematics. It was, in fact, a boarding school, where all expenses were paid by the government. Here students learned mathematics beyond the normal curriculum and were given problems to solve of the order of difficulty of the IMO. Conditions were pleasant, and there was every reason for the students to wish to remain. We were told that, as the students became older, some would be removed from such schools and placed either in a regular school or perhaps in a vocational school.

Many methods are used to train and to encourage outstanding students. In some countries, schools may have a mathematics club in which student members may join in solving problems set by the teacher-organizer. Lectures may be given by speakers from local colleges. Students, too, will be encouraged to talk about some subject of their own choosing.

In some countries, local contests, involving two or three neighbouring schools, are organized, such as those involving the Bronx High School of Science, Walton High School, De Witt Clinton High School and Yeshiva University High School. These were held at each school in turn and provided social contact as well as mathematical experience. These contests add considerably to the interest of the work.

Some colleges hold their own contests. Examples in the vicinity of New York are Stockton College and Rutgers University. But many schools in the United States provide such contests. Elsewhere, there may be regional contests. Examples of these may be found in the USSR and in the United States, where such states as Texas, Wisconsin and California hold state contests.

At this point, it is perhaps relevant to mention a contest that is peculiar to the United States, but which seems to be a very valuable means to develop mathematical ability. This is the American Region Mathematics League (ARML) which is now almost a national contest. Students taking part make their way to a central college, sometimes travelling over 1000 miles to do so. There they indulge in mathematical games. Some problems are for individuals to solve. Some for teams to solve. These include a difficult or 'power' question for teams to work on. Then there are 'relay' problems for school or state teams to tackle. Here each team member has a problem which requires using the result of the previous student to solve. Excitement is high, and the continued growth of ARML testifies to the popularity of this activity.

Periodicals and reference material

It is commonly believed that students learn most or all of their mathematics in the classroom. This is simply not so. Students learn much from each other as well as from reading, lectures, and from other out-of-school activities. In many countries, periodicals form part of the gifted student's sources of knowledge. The USSR, for example, has its *Kvant*, Canada its *Crux Mathematicorum*, Hungary has *KöMal*, and so on. As far as possible, every country should try to provide a mathematical periodical for the use of its students. Again, it is useful to compile a list of relevant books on various subjects. There are many such publications in such countries as the USSR, the United States, the Netherlands and the United Kingdom. The most complete of these is the *New Mathematical Library*, published by the Mathematical Association of America. Some particular texts should also be available on algebra, geometry, trigonometry, calculus and other special topics. These should be in the school library for students to turn to for information and learning. If the school library is inadequate, a local college might perhaps allow selected school pupils to use its library for reference and supplementary reading.

By contacting colleagues in other countries, teachers could obtain copies of problems set in previous contests, especially those set in IMO contests. Such problems can be used to challenge students. The number of countries which issue such collections is growing year by year. Moreover, these collections usually include solutions which other students can study, consider and, perhaps, improve upon.

With so large a number of countries involved in the IMO as well as with their own local contests, it is impossible to be as specific as one would wish to be. It is hoped that what has been presented here will induce interested countries to contact friendly neighbours to learn more. There is, however, one final remark to make. It concerns the supply of contestants. It is often said that because the United States has so large a population, it has a larger pool of students to draw on. The example of Hungary, however, with its much smaller population, is one for us all. In that country, all the media unite in presenting mathematics in a form that will interest the student. It is a matter of motivation. In some countries, soccer dominates the television, in others it may be baseball or cricket. In Hungary, it can be mathematics. It is worth remembering that a child will try anything, provided it is fun and challenging. Educators know this from practice, and educationists might well consider the implications. In *Sylvie and Bruno*, one of Lewis Carroll's stories, the heroine says 'I believe a really healthy boy would thoroughly enjoy Greek grammar if only he might stand on his head to learn it.'

Lewis Carroll was a discerning teacher.

4. National mathematical olympiads in Vietnam

Lê Hải Châu

In order to nurture pupils with special gifts, those whose flair is for mathematics are identified, and favourable conditions are created to help them improve their skills. Since the academic year 1961-62, the means of selection has been the national mathematical olympiads organized by the Ministry of Education. These competitions epitomize a strategic characteristic of Vietnam and they have been approved at all the educational levels, from local to central. They constitute a nation-wide movement which unites school staff, teachers and parents in the task of fostering and training the gifted pupils.

Annual national mathematical olympiads have been organized regularly every April for pupils at the end of the first, second and third levels of education. The selection of competitors is a four-stage process.

The first stage amounts to talent spotting. From the start of the school year, teachers of the first, second and third levels of education (now the basic school and the secondary school) are expected to look out for and to foster pupils with special gifts in mathematics.

During the second stage, these pupils are tested at the ward or at the district levels. Competitions are organized by the Department of Education in each ward or district, the objective being to make up teams of gifted pupils in mathematics, representative of the first and second levels of education. Later, these teams are cared for by instructors in extra classes, as the pupils concerned still have to continue their full-time education in school. Pupils who are selected for the extra classes are often more numerous than those who eventually form the local team, as its final membership is determined after further Department of Education testing. In any event, reserves for the eventual team have to be trained to be available.

The third stage involves local testing in towns and provinces. Competitions for first and second level pupils are organized by the provincial Department of Education. The results determine the provincial teams. As in the second stage, the number of pupils selected for the provincial team is more than those who will make up the final team. Pupils in provincial teams also participate in special mathematical classes. In the same way, special training is given to pupils from the third level of education, that is, secondary school students. They are formed into provincial teams and eventually become candidates for the national mathematical olympiads.

These constitute the fourth stage of the selection process. Problems are sent by the Ministry of Education to all provinces and towns at a fixed time. The contestants, five to ten pupils, according to the regulations of the Ministry of Education, are members of the provincial teams.

At the conclusion of the examinations, the solutions are sealed in front of the supervisors and representatives of pupils who are required to put their signatures on the envelopes. All are then sent to the Ministry of Education. Meanwhile, a Central Examining Board is established by the Ministry of Education. Its purpose is to gather experience for subsequent years.

The results of the olympiad, including the list of the winning teams and of pupils, with their marks, are sent to each province and town. Two kinds of prizes are awarded: individual prizes (first, second, third and consolation prizes), which are intended to encourage pupils with an aptitude for mathematics; collective prizes which are intended to encourage the collective efforts of communities to discover and sponsor gifted pupils.

The process of devising problems in the national olympiads begins with the Ministry of Education inviting experienced professors and mathematicians to submit problems whose solution will invoke imagination and creativity, as well as deductive ability, the hall-marks of a gifted pupil.

A selection committee is set up to validate all the submitted problems and choose those which will comprise the eventual test. These are then sent to the Ministry of Education to be printed and dispatched to the provincial examining boards.

The six tasks which comprised the 1975 olympiad, for students in the last year of secondary schooling, illustrate the scope of the mathematics which is called for and the level of difficulty of the problems set.

First day of competition

1. Find all pairs of integers (x, y) satisfying the equation

$$x^3 - y^3 = 2xy + 8.$$

2. Let M be a set of all functions f defined with all integers and accepting the real values satisfying the following properties:

   a) $f(x) \cdot f(y) = f(x + y) + f(x - y)$ with all integers x, y;
   b) $f(o) \neq 0$.

   Find all the functions $f \in M$ so that $f(1) = 5/2$.

3. A rectangular parallelepiped with dimensions a, b, c is cut by a plane which passes through the intersecting point of the diagonals and is perpendicular to one of the diagonals. Find the area of the section so obtained.

Second day of competition

4. Let the greatest common divisor of a, b be (a,b). Prove that with three natural numbers a, b, and m the necessary and sufficient condition of the existence of the natural number n so that

$(a^n - 1) \cdot b$ is divisible by m is:

$$(ab,m) = (b,m)$$

5. Find all real values of parameter a so that the equation

$$16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$$

has four distinct roots forming a geometrical progression.

6. A triangular pyramid $OABC$ has S as the area of the base ABC. Find the volume of the pyramid if each of the heights drawn from the vertices A, B, C is not less than the arithmetic mean of the two sides of the opposite faces of these vertices.

Preparing the contestants. In order to achieve good results in the national mathematical olympiads, and with a view to encouraging both pupils and teachers, the Ministry of Education organizes special mathematical classes and extra-curricular activities in mathematics. The special mathematical classes are located for convenience in the best school of the province or of the town. Here, the gifted pupils are identified, sponsored and groomed for a future generation of mathematical cadres who will meet the demands for national scientific and technological development. In order to join these special classes, pupils have to pass a June examination organized annually by the Ministry of Education. Participants are those who have been members of the provincial mathematical teams of the second level and who were successful in the national mathematical olympiads set for this level. It is a rule of recruitment that attention should be paid to the quality of the pupils, not to the numbers of them.

Instructors of special classes are chosen from among the most experienced and capable teachers and from those who are enthusiastic about training gifted pupils. These criteria are very important because the intructors must pass on the latest and the most up-to-date knowledge and help the pupils to make rapid progress. An instructor of average ability cannot meet this demand.

The content of teaching in these special classes is 'based on the all-round education needed to train the mathematics aptitudes'. The ground covered in mathematics is the same as that in the normal curriculum of the basic and the secondary schools, but it is enlarged, widened and enriched with modern problems, as well as with some of the latest application of mathematics.

The method of teaching seeks to cultivate political, moral and patriotic qualities while inculcating the love of mathematical thinking. It seeks to give training in exact and topical thought, as well as to encourge independent thought and give scope to active, clever and creative ability. At the same time, attention is paid to an all-round education.

Quite apart from the special mathematical classes, all schools in Vietnam organize extra-curricular activities in mathematics. These create good conditions for pupils with an aptitude for mathematics to develop their skills, and they are popular. The extra-curricular activities are held in the classes at the end of the first, second and third levels of education. Twice a month, the pupils discuss some particular subject, exchange experience and organize mathematical evenings, mathematical visits and mathematical games. Prizes are offered and wall-papers published.

Summary. The national mathematical olympiads, the special mathematical classes and the extra-curricular activities exert a harmonious influence on the teaching of mathematics. They help to mobilize the efforts of teachers in the tasks of organizing and classifying pupils and encourage them to pay keen attention to gifted pupils. They encourage the pupils, so that their parents will pay more attention to their care and to creating good work conditions for them. They create, incidentally, a core of good and experienced instructors able to discover and to sponsor gifted pupils as well as a core of promising mathematical cadres for the country's future. And, finally, they encourage the cadres of all schools and educational offices at all levels to bend their efforts 'teach well and learn well', thereby raising the quality of education in mathematics.

Pupils with an aptitude for mathematics are helped to develop learning habits. Their sense of purpose and their attitudes to learning as well as their passion for mathematics are strengthened; they learn to respect both theory and practice, to learn seriously in class and to learn new lessons while revising old lessons and to develop and implement a reasonable learning plan. They also develop the habit of dipping into reference books so as to enrich their knowledge. Finally, they learn to apply mathematical knowledge to living conditions.

Thanks to the lead given by the Ministry of Education, many Vietnamese students have acquitted themselves well in the international mathematical olympiad. The first team took part in 1974; four of the five participants won prizes, one first prize, one second prize and two third prizes. This initial success gave a great boost to the mathematically inclined in the country as a whole. As a consequence, from 1974 to 1985, at nine subsequent international mathematical olympiads, Vietnamese teams won forty-eight prizes, including five firsts and seventeen seconds.

Part II

Mathematics and the media

5. Broadcasting and the Open University of the United Kingdom

Frank Lovis

The proper use of radio and television in teaching mathematics is of great importance for a distance-learning institution such as the United Kingdom's Open University. This is particularly so because the Open University was required, from its beginning, to make all possible use of modern educational technology. Since in considering radio and television it would be pointlessly restrictive to discuss only open-circuit transmission, the place of audio and video-cassettes in mathematics teaching will also be included in this chapter.

The Open University makes much use of open-circuit transmission, which has been made an exceptionally cheap medium for it to employ. No charge is made for the air time and the Open University pays only for the engineering costs of transmission. These broadcasts provide for the distant teacher the great advantage of gaining entrance into the student's own home. They have, however, created the anomaly that, since a high proportion of the British Broadcasting Corporation's (BBC) viewers tune in, at some time or other, to an Open University programme, the Open University is best known for its television programmes, although these actually occupy less than one-twentieth of the student's study time.

In one respect, the Open University provides a perfect situation for judging the effectiveness of broadcasting in the teaching of mathematics. It has no direct control over its students, who are, in the main, mature and busy people. So programmes which are assessed by the first students of a course as being of little help in passing that course will be largely ignored in the second and successive years of its presentation. Conversely, it is reasonable to assume that a repeated rise in the proportion of listening/viewing students for a series of programmes indicates that those programmes are found to be useful. Not everyone will approve of such a pragmatic approach to the assessment of success in education, but in the case of such an expensive resource, it is ultimately justifiable. There can be no point in putting effort into producing polished programmes if the students choose not to use them. To the Open University student, a successful programme is simply one from which he learns enough to justify (in his own eyes) the time and effort which he spends on it. From its beginning, the Mathematics Faculty of the Open University has been confident that the television provides a valuable component for many of its courses. In contrast, it has been disappointed by the lack of enthusiasm shown by students towards its radio programmes. It thus seems sensible to tackle this problematic area first.

Open-circuit radio programmes

After much thought and research, the Mathematics Faculty of the Open University has concluded that the straight-forward radio broadcast is not a suitable medium for teaching mathematical concepts. Nor does the provision of a written transcript, or even the ability to record and repeat the talk, help very much. Such programmes are an inadequate substitute for good written material and, if such material exists, the addition of live radio makes no significant addition to its effectiveness. So the Foundation Course in Mathematics (M101) contains no such programmes and, in fact, uses live radio in just one way, a way which is adjudged successful because the proportion of students using these programmes is rising from year to year.

The M101 radio broadcasts have two essential features: they are entirely tutorial in nature and they require the student to prepare carefully for the broadcast. Not merely must the student read the written material linked with the broadcast and carry out carefully specified tasks before listening to it, but this work must often be done within a specified structure. The broadcaster thus has a very precise picture of the current mathematical state of his audience. To illustrate, we here reproduce these programmes[1], two from an early stage in the course and two from nearer its end.

Radio programme 1: (broadcast 0635, Friday 21 February, repeat 0700 Sunday 23 February)

1. The bisection method

In this programme we will be looking at the bisection method for finding solutions of equations. The equation we will consider is:

$$x^3 - 6x^2 + 9x - 1 = 0$$

$$\text{Nested form: } ((x-6)x + 9)x - 1 = 0$$

Before the programme, please fill in Table 1. This table will illustrate that there are three roots of the equation in [0, 4].

1. The materials presented below are reprinted with permission from The Open University (1982, 1985, 1986).

Table 1

Value of x	Corresponding value of $x^3 - 6x^2 + 9x - 1$
0	
1	
2	
3	
4	

In the programme we will discuss how to use the bisection method to find approximations to two of the roots, with maximum possible error 0.2; and we will also be giving advice on the easiest way of filling in the following tables.

Table 2

n	a_n	b_n	x_n	Is $x_n^3 - 6x_n^2 + 9x_n - 1 \geq 0$?	x_n replaces a_n	x_n replaces b_n	Maximum possible error

Table 3

n	a_n	b_n	x_n	Is $x_n^3 - 6x_n^2 + 9x_n - 1 \geq 0$?	x_n replaces a_n	x_n replaces b_n	Maximum possible error

Radio programme 3 (broadcast 0635, Monday 24 March, repeat 0700, Sunday 6 April)

Transforming graphs

The tutorial in this programme will be about scalings and translations, and quadratic and cubic functions. Before the programme we would like you to attempt Questions 1 and 2 below.

Question 1

The graph of $y = 4x^3 + 12x^2 + 8x + 8$ is transformed using the y scaling

$$(x,y) \mapsto (x, \tfrac{1}{4}y)$$

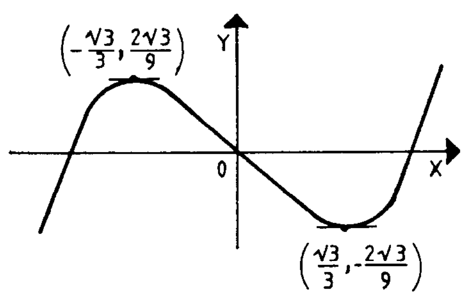
followed by the translation

$$(x,y) \mapsto (x + 1, y - 2)$$

Find the equation of the transformed graph

$y =$

During the programme we shall refer to the following:



One sequence of transformations which takes the graph of $y = x^3 - x$ to the graph of $y = 4x^3 + 12x^2 + 8x + 8$ is

$(x,y) \mapsto (\quad , \quad)$ (A)

followed by

$(x,y) \mapsto (\quad , \quad)$ (B)

$\left(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9}\right) \mapsto (\quad , \quad) \mapsto (\quad , \quad)$ (C)

Question 2

(a) The graph of $y = x^2$ is transformed by using the y scaling $(x,y) \mapsto (x, \mu y)$ followed by the translation

$$(x,y) \mapsto ((x + \alpha, y + \beta).$$

Show that the equation of the transformed graph is

$$y = \mu (x - \alpha)^2 + \beta .$$

(b) The completed square form of $3x^2 + 6x + 8$ is

During the programme we shall discuss the graph of $y = 3x^2 + 6x + 8$.

Radio programme 13 (0635, Saturday 17 August)

Complex numbers and differential equations

There will be two tutorials in this week's programme. The first will be on complex numbers and the second on solving differential equations.

(i) Before the programme we should like you to put the complex number $1 + i$ in polar form.

(ii) Before the programme we should like you to differentiate the following functions.

(a) $u = \sqrt[3]{\left(\frac{3t^2}{2} + K\right)}$ (b) $u = 1 + Ae^{t^2/2}$

where K and A are constants.

(iii) During the programme we shall discuss the solutions of the following differential equations:

(a) $\frac{du}{dt} = \frac{t}{u^2}$ (b) $u - \frac{1}{t}\frac{du}{dt} = 1.$

Radio programme 15 (0635, Saturday 14 September)

Revision 1

In this programme we shall continue our revision for the examination. Before the programme we should like you to think about how you would tackle the following four questions.

(i) Two variables x and y are related in such a way that the graph of $\log_e y$ against x is a straight line passing through the points $x = 0$, $\log_e y = 2$ and $x = 1$, $\log_e y = 5$. Find the function f for which

$$y = f(x).$$

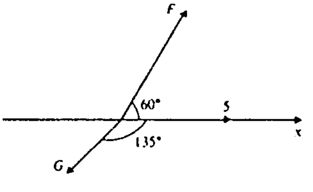
(ii) Let H be the set of matrices

$$H = \left\{ \begin{bmatrix} a & c \\ b & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc \neq 0 \right\}.$$

(a) Show that H is closed under matrix multiplication.

(b) Show, by providing a counter-example, that H is not closed under addition.

(iii) The diagram below shows three forces acting at the same point on a body which is at rest. One force has magnitude 5 kg wt and its direction is that of the x-axis. The other two forces with magnitudes F and G kg wt respectively are as shown in the diagram below. By resolving the forces in a suitable direction, or otherwise, find F.



(iv) Prove, using the method of Mathematical Induction, that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for all integers $n \geq 1$.

Additional Notes

$$\begin{bmatrix} ap + cq & ar + cs \\ bp + dq & br + ds \end{bmatrix} \quad (1)$$

$$(ap + cq)(br + ds) - (bp + dq)(ar + cs) \quad (2)$$

$$(ad - bc)(ps - qr) \quad (3)$$

Note that the earlier programmes rely much more on carefully structured work by the student. At that stage, the habit of listening to the programmes is still being nurtured, whereas, by Programmes 14 and 15, the audience is much more stabilized and is looking ahead to the final examination. Note also the distinctly anti-social timing of the broadcasts: although the use of a tape-recorder and timing device is not actually assumed by the programme's structure, it must be anticipated that most students will not actually listen to the broadcast as it goes out.

Here then is one type of broadcast which definitely does help in teaching of mathematics. But it demands a high price. It needs to be carefully aimed at a point of the course which is known to create general difficulty. Such problem areas cannot always be foreseen by the teachers: they have to be deduced from the actual experience of the students. Hence the first version of the broadcasts will certainly not be the last. Moreover, a lot of time and effort is needed to make such broadcasts and this has come from teachers with a personal gift for radio and with suitably pleasant and energetic broadcasting voices, minority gifts, indeed. (In theory, of course, one could use professional broadcasters, but the Open University has not generally found this successful in tutorial programmes. Somehow, it works better when the voice belongs to someone who knows the course intimately.)

What does such a tutorial programme gain from being broadcast, as opposed to being supplied as an audio-cassette? The answer is nothing; the audio-cassette is altogether better. The Open University's radio broadcasting arises from an economic decision and, in fact, the Open University does assume that every student has a tape-recorder and can afford to buy blank tapes, as necessary. It is simply cheaper to the Open University for the programmes to be broadcast if the number of students on the individual course is more than about 500. Below this figure, we supply recorded audio-cassettes. For other countries, this economic cut-off figure will vary greatly, but it will nevertheless exist. It is, however, obvious that tutorial programmes, which must be made on the assumption that they will be heard only once, will be able to cover less ground and in a shallower way.

OTHER AUDIO-VISUAL COMPONENTS

The next section of this chapter will examine the other audio-visual components of M101. But, before doing so, an explanation is needed of the reasons why this particular mix was chosen. A rudimentary analysis of the stages in mathematical learning will be helpful. We shall therefore consider three stages:

1. *Exposure*: the first meeting with new material.
2. *Experience*: in greater depth and detail.
3. *Mastery*: of either technique or concept.

It has already been made clear that in M101 we restrict live radio broadcasts to Stage 3. Where, then, should audio-cassettes and television be used?

The audio-cassette has a number of basic advantages:

The Freedom to stop/start and repeat (including, most importantly, freedom to stop and think).

The Freedom both to listen and do something else: i.e. look at printed material or write notes.

The presence of an energetic, friendly and helpful voice (significantly less formal than print).

Freedom of choice of time and place to do the work.

The tape can be accompanied by carefully designed sections of the printed material.

(Kern and Mason (1977) give an interesting analysis of the use of audio-cassettes in M101).

Tape has two significant disadvantages. First, its style of presentation cannot be copied from other traditional media. Consequently it makes demands on the creativeness of the teachers who wish to use it. Second, the student simply cannot consume this method of teaching for a long period of time.

The Open University's television programmes have always had the huge advantage of being produced by the BBC. This has guarenteed a professional standard of production. And, since they are transmitted on the national channels, there has been an insistence on a high standard of content and presentation. It also, alas, means accepting something of a strait jacket. For we really do not use television, but television programmes, of 25 minutes duration, neatly packaged with smart introductory and valedictory credits, and fitting more or less within a mould. The professionally made video-cassette (analogous in its flexibility to the audio-cassette) remains in its infancy and is currently too dicey to use for courses, such as M101, with a large body of students (about 3,400). Very few people can actually continue to concentrate on a television programme for 25 minutes. This means that its potential for teaching is much less than might at first be supposed. Moreover, the student who is concentrating on the pictures and the commentary cannot be expected to 'think' at the same time. Mathematical television programmes are still being made today (though not by BBC/Open University Production) which pack the screen with symbols and then proceed to manipulate them much too quickly. Such teaching simply cannot be effective. The knowledge that television is an expensive medium must not tempt one into cramming a programme with content. There must be just a small amount of teaching, carefully signposted, which will ideally bring a difficult concept to life.

It must be recognized, however, that television programmes have an impact of their own, through what has been succinctly described as 'the reverence for television in our culture' (Mason, 1979). Their moving pictures, combined with rapid, effective cutting, can give a quick, dynamic impression of a new concept, which is unobtainable through any other medium. Their undeniable ephemerality can even be turned to an advantage, if they are used at the right stage of the teaching.

To return to our three stages of learning: *Exposure* should be fast, friendly and encouraging, qualities which can be readily incorporated in a television programme. But the programme cannot usefully stand on its own: it must be linked with written material which also introduces the new concept. And the style of this written material must itself be appropriately light so as to preclude the student from mistakenly thinking that he is already in Stage 2 and must consequently understand every word. Audio-cassettes are not appropriate to the stage of *Exposure*; 'stop-start' is precisely what is not wanted. It is in Stage 2, *Experience*, that audio-cassettes are particularly useful. *Experience* in mathematics teaching usually means posing structured questions and setting worked examples, to which the friendly voice on the cassette can give motivation and momentum. In the case of an elusive concept, a television programme can be used at Stage 2 to illustrate it vividly, using, perhaps, computer graphics to show movement, or developing a computerized simulation. But at this stage also, the effectiveness of the pictures must not be lost in a mass of detail. If detail is essential, it must go into print, not on to the screen. M101 does not use either audio-cassettes or television in Stage 3, *Mastery*. In general in mathematics, *Mastery* must mean doing, i.e. routine practice, when the student is ready for it. Some students will need remedial help if they are to achieve *Mastery*, and the tutorial radio programmes seek to provide this.

The integration of the audio-visual components in the Foundation Course

Having considered the individual elements of the audio-visual component, it is time to see how they are all integrated into the total teaching package of M101. To do this, we shall examine Block II of that course, a set of five units entitled Functions and Numbers. Each unit starts with a diagram (see below). This shows how the printed material, the audio-cassettes and the television programme depend on each other. The versatility of the arrangements is noteworthy: the television programme can come before, after, or in tandem with the audio-cassette. The latter can itself be used all at once or in instalments (each audio-cassette runs for a total of 20 minutes). Note that the fifth unit, *Review*, does not have an audio-cassette. At this stage, the student is concerned only with Stage 3, *Mastery*.

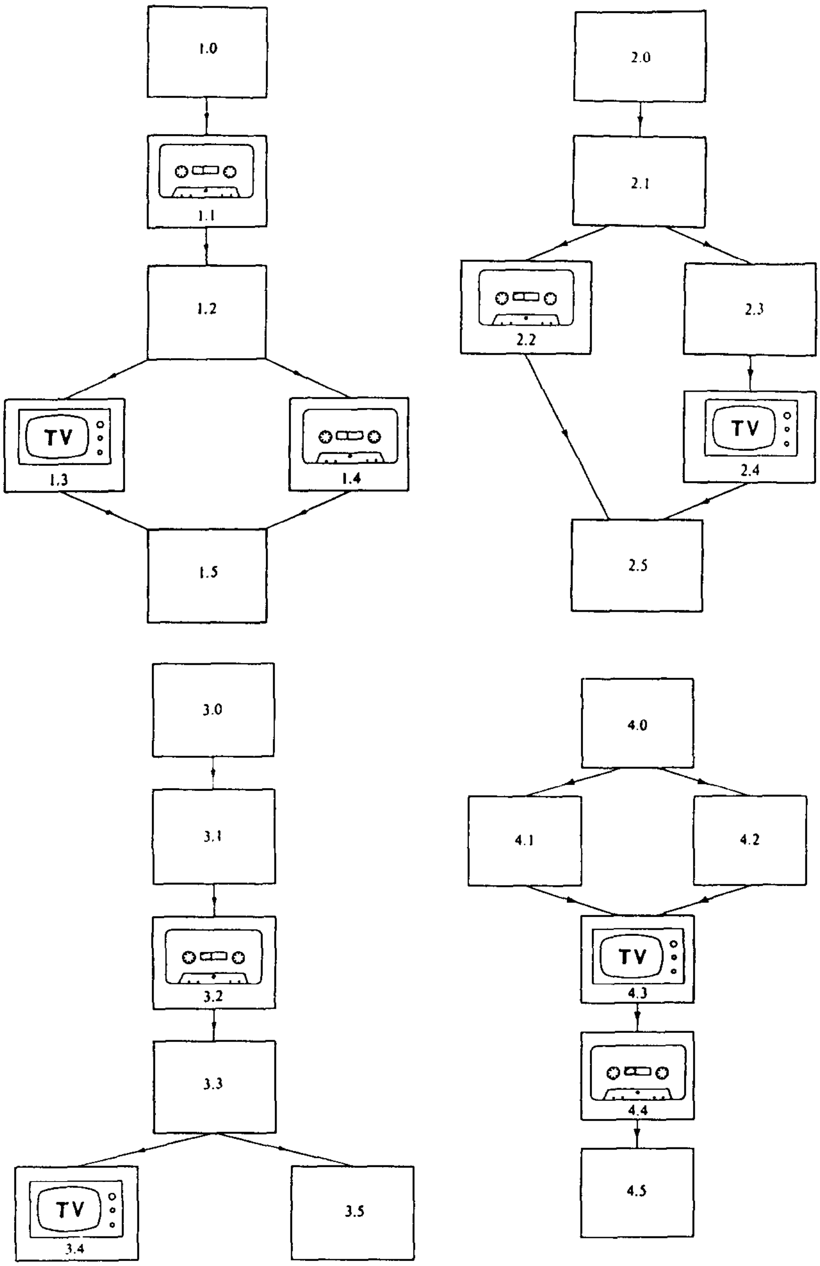
M101. The components of Block II. 'Functions and numbers'

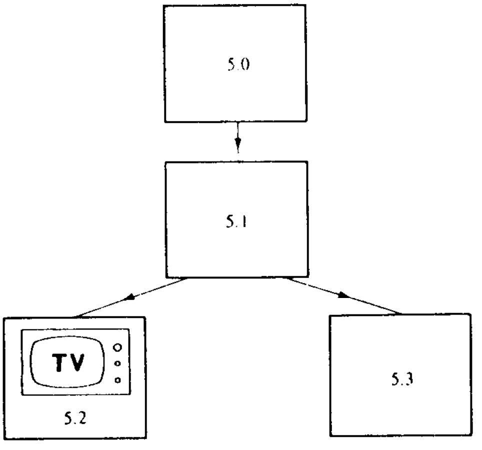
Here now is the introduction to the first use in this Block of an audio-cassette.

1.1 Graph plotting and transformations I

In this section we ask you to plot the graphs of some simple functions, and investigate the result of applying various transformations to some of the resulting graphs.

Before you begin this section, you should remove the transparent sheet from the centre of this unit and cut it along the dotted lines to produce eight numbered overlays. You will need Overlays 1-5, a calculator and a pen or pencil. As you are sometimes required to draw two graphs on the same graph paper, you may wish to use two different coloured pens. (Overlays 6-8 will be used in Section 1.4.)





Dependence of the sections in each of the units

As you see, no fewer than eight overlays are to be used in this Unit. They are bound in with the printed material and have to be cut out by the student. The instructions are followed by fourteen 'frames', of which the first two are shown below. These the student completes as specified on the tape. The audio-cassette is divided into sections of arbitrary length, each of which ends with a musical jingle. These serve as a signal to the student to stop the tape and carry out the tasks which he has just been given.

1 Complete the following tables and use the results to plot the graphs of

(i) $y = x^2$;

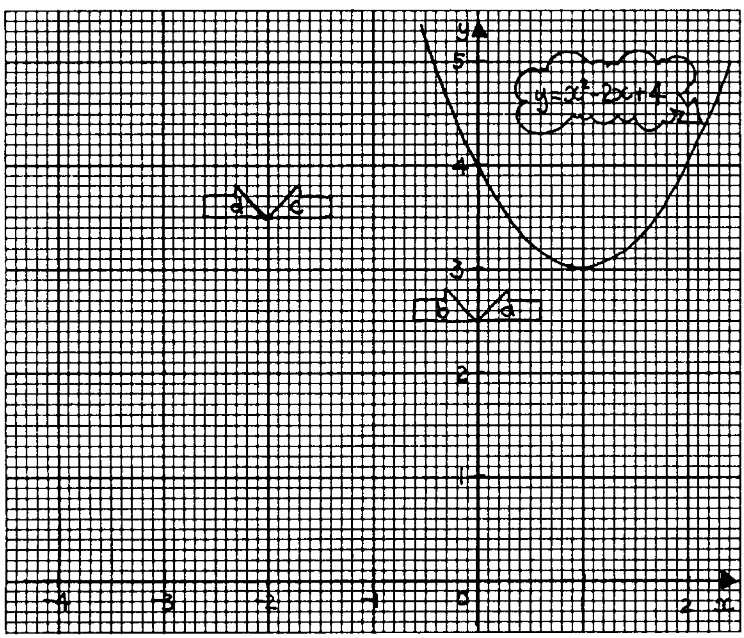
(ii) $y = x^2 + 4x + 5$.

(i)

x	−2	−1·5	−1	−0·5	0	0·5	1	1·5	2
x^2				0·25					

(ii)

x	−4	−3·5	−3	−2·5	−2	−1·5	−1	−0·5	0
x^2+4x+5		3·25							



2 The translation which takes the graph of $y = x^2$ to the graph of $y = x^2 - 2x + 4$ is

$$(x, y) \mapsto (x + \square, y + \square).$$

We continue to look at the structure of Unit 1 of this Block. The student now works through the printed material of Section 1.2 and can then either go to Section 1.3, the television programme, or to Section 1.4, the second part of the audio-cassette. Such alternative branching is not always possible, but when it is possible, it can help to avoid a hold-up in the student's work arising from the inflexibility of the television transmissions, which cannot be arranged to come at exactly the same point in the week's work in every year of the running of the course.

There is a substantial piece of Pre-programme Work, part of which is shown below. In all, five problems are set, their solutions being placed at the end of the unit.

1.3 Classifying cubics

Pre-programme work

In the previous section we used transformation to give some insight into the graphs of the *quadratic functions*

$$x \longmapsto ax^2 + bx + c \quad (x \in \mathbb{R}),\ a \neq 0.$$

In this section we shall undertake a similar programme for the *cubic functions*

$$x \longmapsto ax^3 + bx^2 + cx + d \quad (x \in \mathbb{R}),\ a \neq 0.$$

In Section 1.2 we concluded that all quadratic graphs are essentially the same as that of $y = x^2$, in the sense that any quadratic graph can be obtained from the graph of $y = x^2$ by means of scalings and translations. In our investigation of cubic graphs we consider whether a similar result is true for these: i.e. are all such graphs essentially the same as that of $y = x^3$?

For quadratics we discovered that any quadratic function with leading coefficient 1 (i.e. $a = 1$) could be obtained from $y = x^2$ by means of a single translation. To begin the investigation of cubic graphs, we ask you to investigate whether the corresponding statement is true for cubic graphs.

Problem 1.3.1

(i) Find the equation of the graph obtained by translating the graph of $y = x^3$ by $(x,y) \longmapsto (x + \alpha\ ,\ y + \beta\ )$. (You may find it helpful to expand using the Binomial Theorem.)

(ii) Show that $y = x^3 + 3x^2 + 3x + 2$ is a translated form of $y = x^3$. (Hint: Equate coefficients as in Solution 1.2.5.)

(iii) Show that $y = x^3 + 3x^2 + 4x + 2$ is not a translated form of $y = x^3$.

(iv) Decide whether or not $y = x^3 + 3x^2 + 2x + 2$ is a translated form of $y = x^3$.

Solution 1.3.1 shows us that, unlike quadratic functions, a leading coefficient of 1 does not mean that we can achieve our objective by applying a single translation. Let us, therefore, explore what happens when we have a combination of transformations.

Problem 1.3.2

(i) Find the equation of the graph obtained from $y = x^3$ by first x scaling by a factor λ , then y scaling by a factor μ (λ and μ not equal to zero), and lastly translating by $(x,y) \longmapsto (x + \alpha\ ,\ y + \beta\ )$.

(ii) Show that $y = 2x^3 - 6x^2 + 6x + 5$ can be obtained from $y = x^3$ by an x scaling and a y scaling followed by a translation.

(iii) Show that $y = x^3 + 3x^2 + 4x + 2$ cannot be obtained from $y = x^3$ by an x scaling and a y scaling followed by a translation.

(iv) Decide whether or not $y = 2x^3 + 6x^2 + 4x + 2$ can be obtained from $y = x^3$ by an x scaling and a y scaling followed by a translation.

As Solution 1.3.2 indicates, the situation for cubic graphs is rather more complicated than that for quadratic graphs. However, all is not lost, as the television programme will show. In the programme we shall use the solutions to Problems 1.3.3-1.3.5.

Next comes the *Review of the Television Programme*, which is in considerable detail. Ostensibly, this *Review* exists to help any student who lost track of the programme or who cannot remember its content clearly, but it also serves to 'rescue' any student who, willingly or unwillingly, did not see the programme at all and who must nevertheless be steered through the rest of the Unit's work. Clearly, the presence of this detailed *Review* increases any temptation which a student might feel not to bother with the television programme of the unit and, equally obviously, it obliges the academics to ensure that the television is a valuable component. Here is the first of the three pages of the *Review*:

Review of programme

The programme began by reviewing the effect of translations and scalings on quadratic functions. In particular, it considered the quadratic function

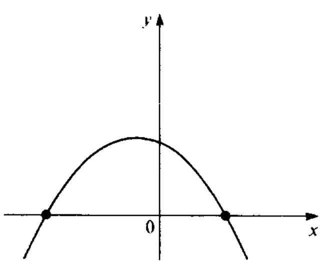
$$x \longmapsto 4x - x^2 \quad (x \in \mathbb{R})$$

whose rule is the same as that of the trajectory function in the television section of Block I, Unit 4. By completing the square for the quadratic expression $4x - x^2$, we saw that the graph of $y = 4x - x^2$ could be obtained from the graph of $y = x^2$ by first reflecting in the x-axis and then translating by $(x,y) \longmapsto (x + 2, y + 4)$.

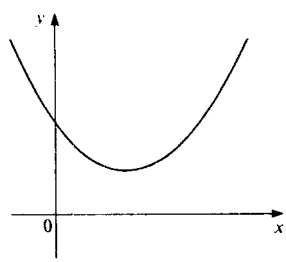
Problem 1.3.6

Write the above expression $4x - x^2$ in completed square form, and use this form to verify the above statement about the graphs of $y = 4x - x^2$ and $y = x^2$.

The fact that any quadratic expression $ax^2 + bx + c$, $a \neq 0$, can be put into completed square form, shows that the graph of $y = ax^2 + bx + c$ ($x \in \mathbb{R}$), $a \neq 0$, can be obtained from the graph of $y = x^2$ ($x \in \mathbb{R}$) by a y scaling followed by a translation. It was then shown that the graph of $y = ax^2 + bx + c$ was related to the corresponding equation $ax^2 + bx + c = 0$, since the solutions of this equation correspond to the x-values for which $y = 0$; in other words, the x-values at which the graph crosses the x-axis. So, a knowledge of whether the vertex of the graph was a maximum or minimum, and the value of the function at the vertex, enabled us to determine the number of solutions to the corresponding equation. For example, if we have a maximum at which the function is positive, then the corresponding equation has two solutions; whereas if we have a minimum at which the function is positive, then the corresponding equation has no solutions.



positive maximum value
two solutions

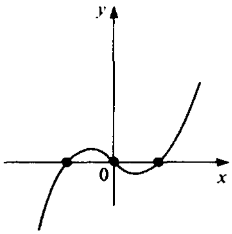


positive minimum value
no solutions

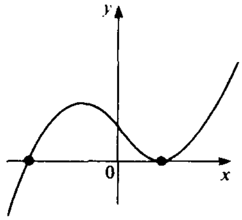
The programme then turned to a consideration of cubic functions and asked the following questions.

1. Could the graph of any cubic function be obtained from the graph of $y = x^3$ by a sequence of scalings and translations?
2. How could we determine the basic shape of the graph of a cubic function from its equation?
3. Given a cubic function, could we determine the number of solutions to the corresponding equation?

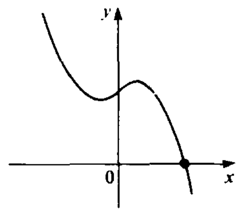
An investigation of cubic graphs led to the hypothesis that there were at least two distinct types. Firstly, those like the graph of $y = x^3 - x$ which had a 'peak' and a 'trough', whose corresponding equation could have either three solutions, two solutions or one solution.



$y = x^3 - x$
three solutions

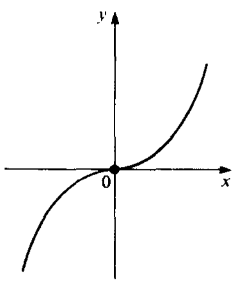


$y = (x - 1)^2 (x + 2)$
two solutions

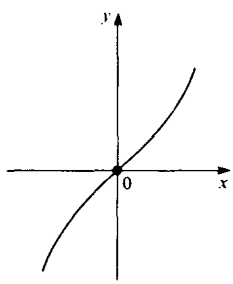


$y = x^3 + x + 2$
one solution

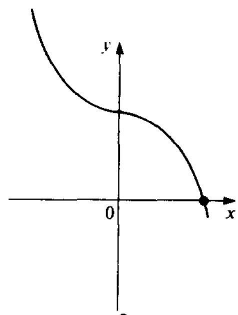
Secondly, those like the graphs of $y = x^3$ or $y = x^3 + x$, which had no peak or trough, whose corresponding equation always had one solution.



$y = x^3$
one solution



$y = x^3 + x$
one solution



$y = x^3 + 1$
one solution

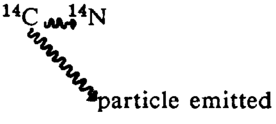
The television component of the Unit then ends with *Post-programme Work*, similar in style and length to the *Pre-programme Work*. This is followed (unless the student has chosen to study it before the television component) by the second section of work with the audio-cassette. This section presents a further nine frames to be completed, in accordance with the instructions on the tape. The concluding section (1.5) uses printed material only. The one remaining component of the unit is the tutorial Radio Programme 3 (see above, pages 45 et seq.), which relates to this unit.

Most of the television programmes of this Block are a direct treatment of part of its mathematical content, but there is one exception in Unit 2. This is now described, in order to show how the course expands its presentation of mathematics whenever the chance occurs. The problem is that of defining a^x for irrational values of x. It is approached via an interesting application of mathematics to the dating of an ancient piece of wood. Here are the relevant pages from Unit 2.

Review of programme

The programme began by describing the archaeologist's problem of dating a piece of wood from an ancient wooden roadway in Somerset. Since evidence from finds nearby is unreliable, a technique is used based on properties of the object itself.

The method described in the programme makes use of the facts that wood, like living or once living matter, contains carbon, and that a small proportion of carbon atoms are radioactive. The programme described the radioactive form of carbon, carbon-14 (written ^{14}C), and its decay into nitrogen: the carbon-14 content of the sample of wood decays with age. This decay of carbon-14 is the basic physical process used to date the wood.



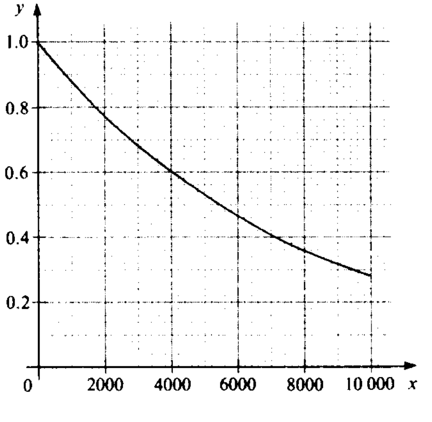
The decay of an individual carbon-14 atom is unpredictable but, when large numbers are looked at, patterns begin to emerge: the programme illustrated this idea by considering dice with five red faces and one white. When a large number of dice were rolled together, about one-sixths of them landed white face up. The number of dice left after each throw was about five-sixths of the number thrown. The behaviour of the dice was modelled by the graph of

$$x \longmapsto 100 \times \left(\frac{5}{6}\right)^x.$$

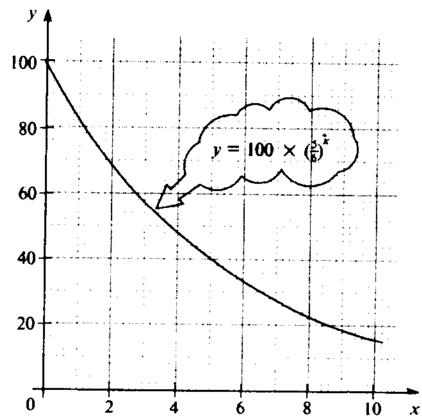
In large numbers, carbon-14 atoms behave similarly, except that the proportion decaying is 1/8000 per year.

In the programme, we modelled the decay of carbon-14 by

$$x \longmapsto \left(\frac{7999}{8000}\right)^x.$$



The general shape of the graph was similar to the graph modelling the behaviour of dice, but drawn to a different 'time scale'.



[As you saw in the pre-programme work, the graph of $y = \left(\frac{7999}{8000}\right)^x$ is obtainable from $y = (\frac{1}{2})^x$ by an x scaling and so, in some sense, these two graphs are 'essentially the same'. We could adopt the methods used in Solution 2.4.3 to show that all exponential graphs can be obtained from each other by suitable x scalings.]

The programme described the 'time scale' for radioactive decay by taking the time for half the radioactive material to decay - the *half-life* of the substance concerned. For carbon-14, the half-life is about 5545 years because

$$\left(\frac{7999}{8000}\right)^{5545} \simeq 0.5.$$

Having arrived at a mathematical description of ^{14}C decay, the programme went on to the problem of finding out what proportion of the original ^{14}C in the sample of wood was left. In order to understand the technique used, we described the 'life-cycle' of ^{14}C from its creation by cosmic rays in the upper atmosphere to its eventual decay into ^{14}N.

Because ^{14}C is thoroughly mixed with ordinary carbon in the atmosphere (in the form of carbon dioxide), the original tree from which the sample came absorbed ^{14}C atoms in whatever proportion was in the atmosphere when it was growing. Once in the tree, no new ^{14}C is formed (because cosmic rays do not reach ground level with sufficient energy) so the ^{14}C in the tree just decays. The ratio of ^{14}C to ordinary carbon in the atmosphere has been more or less constant over a very long period.

Thus, the original ^{14}C content can be measured by measuring the ^{14}C content of a piece of new wood. The technique described involved taking equal sized carbon samples from the old and new wood and measuring the decays in a given time period. Since the proportion decaying in a year (1/8000) is known, the numbers of ^{14}C atoms in the sample can be estimated.

Because the number of decays is small, very elaborate precautions are taken to exclude background radioactivity.

The measurements made at Harwell gave a figure of 0.6 for the proportion of the original ^{14}C in the sample from Somerset. The age was estimated by solving

$$\left(\frac{7999}{8000}\right)^x = 0.6$$

by using the graph. This gave a value of approximately 4000 years.

A more accurate value was obtained by bisection, but this promptly raised a mathematical problem. The nest of intervals obtained by bisection might have been homing in on an irrational number, and $(7999/8000)^x$ has not yet been defined for irrational values of x. The programme used this idea to produce a definition of a^x for an irrational value of x, say $x = r$, which we can summarize as

1. Select a nest of closed intervals with rational end-points, homing in on r, i.e. a nest

   $$[a_1,b_1] \supseteq [a_2,b_2] \supseteq \dots \supseteq [a_n,b_n] \supseteq \dots$$

   (where $a_n,b_n \in \mathbb{Q}$ for $n = 1,2, \dots$), such that r is the only number contained in every interval $[a_n,b_n]$ for $n = 1,2, \dots$

2. Map the closed intervals $[a_n,b_n]$ by $x \longmapsto a^x$. (The fact that a^x is decreasing for $0 < a < 1$ and increasing for $1 < a$ ensures that the images are again closed intervals which form a nest.)

3. The value of a^r is the number which the nest of image intervals is homing in on.

This technique relies on the following basic property of the real numbers: Given a nest of intervals

$$[a_1,b_1] \supseteq [a_2,b_2] \supseteq \dots \supseteq [a_n,b_n] \supseteq \dots,$$

whose lengths tend to zero, then the nest homes in on a unique real number. In other words, there is precisely one real number contained in every interval $[a_n,b_n]$ for $n = 1,2, \dots$

Once the definition of a^x has been extended to the whole of $\mathbb{R}$, the function

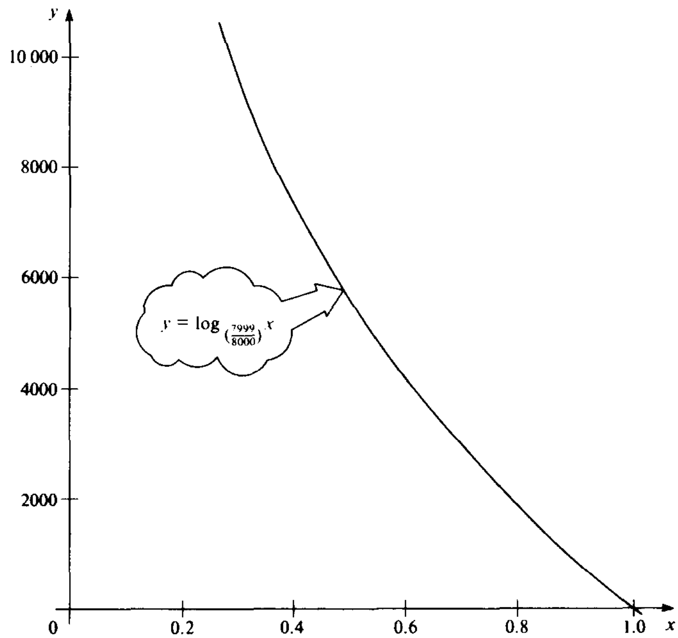
$$\mathbb{R} \longrightarrow \mathbb{R},$$
$$x \longmapsto a^x$$

can be inverted. The programme did this in the particular case of

$$x \longmapsto \left(\frac{7999}{8000}\right)^x \qquad (x \in \mathbb{R})$$

to obtain the corresponding logarithm function:

$$\mathbb{P} \longrightarrow \mathbb{R}$$
$$x \longmapsto \log_{\left(\frac{7999}{8000}\right)} x.$$



We then observed that the solution of the age problem was in finding

$$\log_{\left(\frac{7999}{8000}\right)} 0.6.$$

Since the required logarithm is not available on the calculator, we used the fact that exponential functions are 'the same' except for x scaling to deduce that logarithms are the same except for y scaling. The factor needed to convert the logarithm on the calculator to the one we wanted is approximately

$$-18420$$

so the age of the piece of wood is approximately

$$-18420 \times \log 0.6 \simeq 4086 \text{ years.}$$

Notes on Assumption. As with any use of mathematics and/or physics to solve a real problem, the ultimate justification rests on being able to cross-check on the method.

Tree rings provide one of the basic cross-checks on ^{14}C dating methods. Starting with modern trees, a study of the width, composition, etc., of tree rings has enabled a 'calendar' to be constructed going back some 8000 years. Carbon-14 dating can therefore be carried out on samples of known age to give a check on the accuracy of the method. Such checks indicate that ^{14}C dating is capable of giving good results for objects up to 10 000 years old.

Post-programme work

Problem 2.4.5

Solve

$$\left(\frac{7999}{8000}\right)^x - 0.6 = 0$$

by bisection. Stop when the maximum possible error in your result is 10. Start with the interval [3950, 4100].

Problem 2.4.6

The decimal approximation to $\sqrt{2}$ is

$$\sqrt{2} \simeq 1.4142136 \ldots.$$

which gives a nest of intervals homing in on $\sqrt{2}$:

$$[1,2], [1.4,1.5], [1.41,1.42], \ldots .$$

Find the corresponding nest of intervals homing in on $2^{\sqrt{2}}$. Hence, find an approximate value for $2^{\sqrt{2}}$ correct to 3 decimal places.

In this section we have looked at two main things. First, the role that exponential functions play in solving a practical problem. The scheme we followed,

real problem	→	physical description	→	mathematical description ↓
cross-check	←	interpret solution	←	solve mathematical model

is fairly typical of what happens when mathematics is applied to problems. Secondly, in using bisection to solve

$$\left(\frac{7999}{8000}\right)^x = 0.6,$$

we uncovered a method of defining a^x for irrational values of x and hence obtaining an inverse function for $x \mapsto a^x$ ($x \in \mathbb{R}$).

SUMMARY

The total teaching package of M101 can reasonably be described as intricate, if not complicated. It is the result of running a Foundation Course in Mathematics for some fifteen years. Its structure differs greatly from that which was used in 1971. Since the audio-visual component is expensive both in terms of money and of the time and effort of the academics, it has been treated quite ruthlessly. What could not be seen to be effective has been discarded. A continuously running course is certainly not the easiest field for experimentation and the changes/improvements which are being made to it now come in the form of 'running rewrites' of parts of the course. Eventually, the television programmes will have to be replaced, for they will come to look so old-fashioned that this will get in the way of their teaching. Perhaps, by that time, we shall be able to use video-cassettes, which will bring a new and extremely powerful element to the teaching package. Another medium, in fact, to provide a further considerable opportunity . . . and challenge.

References

KERN, L.; MASON, J. 1977. Non-broadcast Media Technologies at the British Open University. *Education Broadcasting International*, September, pp. 106-11. [New title as from 1982: *Media in Education and Development*, Peter Peregrinus Ltd, PO Box 26, Hitchin, Herts. SG5 1SA, United Kingdom.]

MASON, J. H. 1979. Which Medium, which Message? *Visual Education* (London, National Committee for Audio-visual Aids in Education), February, pp. 29-33.

THE OPEN UNIVERSITY. 1982. *Mathematics Foundation Course. Block II Functions and Numbers*. Unit 1: *Functions and Graphs*; Unit 2: *Functions and Inequalities*. Milton Keynes, Open University Press.

———— . 1985. *M101 Stop Press 2 (1985)*; 1986: *M101 Stop Press 1 (1986)*. Milton Keynes, Open University Press.

6. Distance education in mathematics

Gordon Knight

Distance education in New Zealand

New Zealand has a population of some three million people living in an area approximately equal to that of the British Isles. The country is long and narrow, about two thousand kilometres from North to South, comprising two main islands with several smaller islands off the coast. The education system is national rather than regional; distance education began in response to the needs of those New Zealanders, both children and adults, who do not live within reach of educational institutions which they might otherwise attend.

In 1922, the New Zealand Correspondence School was formed to provide primary schooling (ages 5 to 12) for children living in remote areas (Tate, 1981). And in 1929, courses at secondary school level (ages 13 to 18 years) were added. Over the years, the school has grown to become one of the largest educational institutions in the country.

Distance education at the tertiary level is undertaken principally by two institutions: Massey University, through its Centre for University Extramural Studies, and the New Zealand Technical Correspondence Institute. As the names suggest, Massey University provides opportunities for university study, while the Technical Correspondence Institute offers vocational courses directed towards national qualifications issued by various statutory bodies.

Mathematics courses are provided by each of the institutions, but this chapter offers some reflections and suggestions based on the author's sixteen years of experience of teaching mathematics at a distance in the Department of Mathematics and Statistics at Massey University.

Distance education in mathematics at Massey University

In 1960, distance education in university subjects began with the establishment of a branch college of Victoria University at Palmerston North.

The move was vigorously opposed by many academics and the early days were characterized by the need to prove the value of the system (Owens, 1985). From the beginning, an integrated mode has been used in which the distance learners are kept parallel with a group of on-campus students; the performance of the two groups is thus available for comparison. The same members of staff teach and assess both sets of students, who also take the same examinations and qualify for the same degrees and diplomas.

Although this policy was adopted to safeguard the standards of the distance courses, there are many who would now argue that the benefits to the internal teaching are greater than those to the distance teaching. The discipline required of a teacher to prepare material for distance teaching has very significant effects on the internal classes.

Initially, staff were appointed in English, education, history, geography and mathematics. The sole mathematics lecturer was something of a character. According to Owens (1985) he slept for parts of the day, gave his lectures, then worked in the deserted building through the night. As he marked assignments, he would carol his favourite operatic arias to himself, sometimes taking off for a quick bite at the famous Palmerston North pie cart.

From these early days, when about forty distance students were enrolled in the two first-year courses offered, the work has grown, so that in 1985 the twenty members of the Mathematics and Statistics Department taught fifteen courses to over 1,100 students. The distance courses now cover a full range of mathematics and statistics from first to third year and, already, a number of students have completed Bachelor's degrees, majoring in mathematics. The success of some of these students has stimulated the department to plan to make post-graduate study and research available to them.

As one might expect, the students of mathematics at a distance vary considerably in age, background and motivation. The youngest was a boy of 10, David Tan of Christchurch. Under the tutorship of his father, his exceptional mathematical talent had been developed to the point where he passed the New Zealand University Entrance Examination in mathematics while he was still in the early primary school classes. University regulations prohibited the enrolment of students under the age of 16, and also of those who were still enrolled full-time at school. However, a special case was made for David Tan and he completed his first mathematics paper in 1978 as a distance student and gained an 'A' pass. He later obtained a first class honours degree in mathematics from Canterbury University at the age of 15 and is currently completing a Ph.D. At the other end of the scale, there are many students who enjoy the stimulation of university study later in life or who seek a change in employment prospects which university qualifications will give them. According to Owens (1985), one such student writes:

> To be able to start again to go for a degree in middle life and in the middle of living is a way of taking charge of a life increasingly given over to circumstance. It is a victory immediately one enrols, a type of reversal of fortune that gives what middle-aged joggers are looking for

before their knees give out and there's the beauty of it: the knees of the mind don't give out.

The mathematics courses

In mathematics, the distance courses are based very largely on textual material. Typically, a course will depend heavily on a prescribed textbook. Study guides are prepared. These provide supplementary explanations where such are required. They also give worked examples and suggest exercises. If a suitable textbook cannot be found to cover all the content of the required course, then supplementary material is written. Students complete regular assignments, usually fortnightly, and are given the opportunity for face-to-face contact with lecturers during three-to-five day vacation courses. Initially these courses were compulsory for all students; now they are usually voluntary.

A variety of other means of communications has been tried over the years. Regional meetings of students with tutors have been held, but they suffer from the small concentration of students in any one course in a particular area. More successful is the very strong student support network for distance students devised and operated by the Massey University Extramural Students Society (Williams and Williams, 1985). The network is based on voluntary area communicators and through these, students in the same, or similar, courses help each other.

The telephone has been tried and found wanting as a means of communicating mathematics. It is difficult enough to read without ambiguity a mathematical statement such as

$$[(x - 3)^2 - 5] + 4x = \frac{2x^2 + 1}{7}$$

but to discuss the nature of a graph without using any visual communication is virtually impossible. Consequently attempts at telephone tutorials either on an individual basis or in groups using conference facilities have not proved successful.

A recent development has been the production of audio tapes. These are used by students in conjunction with work sheets which provide the visual communication which is lacking on the telephone.

Students listen to the tape while following the written material. They stop the tape while thay complete an exercise. They then turn on the tape again to check their solution and listen to an explanation. The tapes produced so far have been largely for revision and remedial purposes, but continued developments in this area are likely.

Learning mathematics from textual material

The principal skills which distance students of mathematics require are those of learning mathematics from textual material. In a rapidly changing technological society, these skills are extremely valuable for any student of mathematics. It is most unlikely that the mathematical concepts and techniques which a student learns at school or at tertiary level will be sufficient for the whole of his or her working life. The ability to update one's mathematical knowledge by the private study of textbooks is likely to be an increasingly valuable asset for anyone using mathematics in their work. This may be seen as an argument in favour of distance education based on textual material as against internal study, dependent on lectures and tutorials.

The view of the Department of Mathematics and Statistics at Massey University is that one does not learn mathematics by listening to lectures, by reading textbooks, by talking about it, or by watching someone else doing it. One learns mathematics by being actively involved in the solution of mathematical problems. It will, of course, be necessary to attend lectures, to read, or to discuss before one is in a position to be able to tackle problems. But it is in the active, problem-solving mode that permanent mathematical learning takes place. Managing the learning process, then, involves presenting information, linked to the previous knowledge of the student, together with providing exercises which will reinforce these links and extend and refine the 'cognitive map' (Skemp, 1979) which the student is using.

Cognitive maps

The concept of a cognitive map is useful in this and many other discussions of mathematics learning. Skemp suggests that the conceptual structures which a learner brings to bear on a particular problem may be represented by something resembling a road map. Within this map we may identify two states: a present state *P* representing the stimulus which caused the retrieval of the structure from memory, and a goal state *G* representing the solution to the problem. Understanding consists of realizing both *P* and *G* within an existing framework and making the connections between them. This is illustrated in Figure 1.

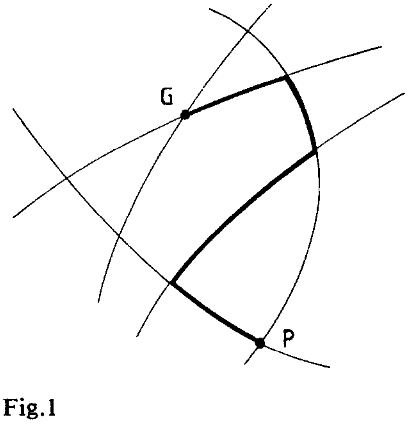


Fig.1

A student faced with the problem 'Solve the equation, $x^2 - 3x = 4$' will retrieve a map which may contain information about factors, 'completing the square', a formula, complex roots, etc.

If this information is correct and sufficiently developed, G may be reached by a number of alternative routes. Failure may occur if the problem does not fit into any existing framework, if the map is not sufficiently developed to ensure G, or if the learner is unable to make the necessary connections. For example, a common error in the solution of quadratic equations is to reason thus:

$$x^2 - 3x = 4$$
$$x(x - 3) = 4$$
$$x = 4 \text{ or } x - 3 = 4$$
$$x = 4 \text{ or } 7$$

In this case, the cognitive map probably contains several of the appropriate features, but the connections between them are faulty.

Using study material to develop cognitive maps

The question that then arises is how distance students in mathematics might best use the course material supplied to them to form the necessary cognitive maps. A few years ago a number of experienced and successful Massey University distance students of mathematics were asked to describe how they approached learning a unit of work from the study guide and textbook. There were, of course, variations in the detail of the answers, but the general pattern was very similar. The following response is typical and quite revealing:

> My first response is just 'picking at' the job to see if any 'meat' happens to fall off the 'bone'. It often does, but sometimes it does not and I must know beforehand to expect this at times. I may try a problem, look at something interesting, or browse through the texts. Any notes I make, and I don't often do this at the first sitting, are in pencil in the study guide. The purpose of the first attack is to obtain familiarity with the topic. If I gain a little more in the exercise, fine, but that is beside the point. I may underline key points in the text; I may reserve this for a time of greater understanding. One must be a little flexible.

This preliminary stage of 'getting to know' or becoming 'familiar' with the topic (sometimes called 'exposure') was mentioned by each of the students in the survey. This is interesting because it is in stark contrast to the way in which the same mathematics is normally presented to internal students in the lecture mode. In this mode, the lecturer may give a brief statement of aims, but the material is usually developed in a logical sequence, the intention being to explain each point in detail as it arises.

In making his 'cognitive map', the distance student may be said to be constructing a sketch map first with the intention of filling in the detail later, while the lecturer may be said to be constructing the complete map one section at a time.

The student would seem, when he talks of 'looking at something interesting' and 'browsing through the texts', to be looking for stimuli which will cause the retrieval of previously established, but relevant, cognitive maps. In 'trying a problem', he is both testing the relevance of these maps and beginning to extend them to make connections with the new material. There is much in common here with Ausubel's (1963) concept of 'advance organizers'.

The student continues his account:

> The second attack is not more positive but equally positive! It appears to be more productive however. Some of the 'meat' is now loose from the 'bone' and my main purpose is to 'pull if off'. It may not all come off, but most should start to move away by now. Problem solving becomes a more important task now, and a real desire to get problems out grips me. I am not interested in note-taking yet, but I may find myself doing some of it all the same. If so, fine, if not, fine too. The key exercise in the second sitting is to 'get into it' and that done I feel quite happy. If unsuccessful I am either overtired and must leave it, or try again until I make some definite progress.

This stage may be thought of as gaining experience, or filling out the detail of the cognitive map. The key role of problem solving is educationally important. It supports the view mentioned previously that one learns mathematics when actively involved in problem solving. The cognitive map is built up by trying out pathways, rejecting those which do not lead in the required direction and reinforcing those which do.

> The third attack is quite easy to make. The assignment must be finished. Unsolved problems must be hammered out. A brief summary must be made for reference purposes. This all fits into place neatly and needs little exposition. I do not make copious notes! I find it better to write down a minimum, and that for reference purposes. An active mind is worth more than an encyclopaedia.

This is the stage of mastering the topic. One can say that the cognitive map is being tested by the completion of the assignment. When this is marked and returned to the student, it may be necessary for him or her to return to the second stage to rectify errors. The recording of a minimum of references may be seen as concentrating on key ideas which will aid the retrieval of the map from memory when required.

The design of course material

The three-stage process of forming a sketch map by browsing, filling in the details by problem solving, and testing the map by completing the assignment, is one which makes sense theoretically and has been found useful by many students. How then may course material be prepared to aid each of these stages?

The character of the presentation of material in the textbook, or notes, is very important in the first 'review' or 'exposure' stage. It invokes the general question of the readability of mathematical text which is a complex one. It has been approached by researchers in a number of different ways (Aitken, 1972; Hater and Kane, 1975; Brunner, 1976; Shuard and Rothery, 1984). The work of Watkin (1979) seems to be particularly useful in the context of helping students to review, rather than study in detail, the material in a text. She found that college students understood the mathematics from textbooks better when the material was written with more common grammatical structures (Ordinary English) rather than in Mathematical English with its characteristic precision. Of course, the mathematical statement of a definition or a theorem is an essential part of the subject and its understanding may well be one of the objectives of the course. However, if key concepts and results are identified and presented in a less formal, intuitive, manner initially, this can greatly assist in the development of the 'sketch cognitive map'. This technique is one which is often used by good teachers in the lecture situation and needs to be incorporated in textual material to help distance learners.

The key role of problem solving in the second stage means that the selection and placement of problems in the study material are critical. It is important to distinguish between the purpose of the questions asked in this section and those which might be asked in an assignment or an examination. In developing the cognitive map, the student will need to try out various paths, reinforce those which are appropriate and delete those which are not.

Carefully graded exercises, worked examples and solutions or hints are required. It is also necessary to provide opportunity for help at this stage through student groups, tutorials or written enquiry slips. Experience at Massey University has shown that students vary enormously in the number of problems they feel they need to solve before they have confidence in their cognitive map. Evaluative questionnaires have been used in many courses to identify sections of material which require more explanation and to determine how long students are spending on the work. It is not uncommon for the time taken by students with apparently similar mathematical backgrounds and abilities to differ by a factor of ten. One student may take two hours and the other twenty on the same section.

The third stage involves testing the cognitive map by completing an assignment which is sent in for marking. Since the objectives of the questions in the assignment are different from those used in the development stage, their character and form may be different (Wynne-Wilson, 1978). It is important to ask questions which will enable the marker to identify any misunderstanding the student may have. The marking of assignments and the comments made on them which are returned to the student are critical at this stage. Harrison (1980) provides an extremely valuable discussion of the role of tutors/markers in distance education mathematics courses, and he points to a number of features of good and bad tuition which he identified in an analysis of marked scripts from the Mathematics Foundation Course from the Open University in the United Kingdom. The positive characteristics include such features as 'friendly tone', 'suggestions for improvement', 'use of references' and 'help with notation'. The negative features include 'failure to correct', 'over-penalizing', 'inadequate comments' and 'unsympathetic response'. The purpose of the responses from markers is both instructional and motivational. Students must be encouraged to re-examine a previously finished section of work and perhaps to go back to the second stage of development and work more examples to correct errors in their cognitive maps. Since they will already be working on the next section by the time they receive their marked work on the previous section, a poor response from the marker is most unlikely to initiate this important step in the learning process.

The future

Textual materials are, and are likely to remain, the mainstay of distance education in mathematics. It is, however, appropriate to consider the possible future role of computers and other audio-visual media in this regard.

There has been a world trend away from using broadcasting by distance learning systems (Bates, 1982) and it is most unlikely that radio or television will in the future be used in an instructional role in New Zealand distance education in mathematics. The relatively small target audience would render it uneconomic, even if it were efficient.

The key to the use of other media, such as computers and video cassettes, seems to be their accessibility to the students. Although New Zealand is by world standards a prosperous country, it is very far from the stage where every student may be expected to have a computer or a video cassette player in his own home. Access could be provided in some of the larger centres, but the spread and sparcity of the New Zealand population, part of the justification for providing distance education, mean that these media are unlikely to be used in anything but a supporting role for some time. The great variety of makes of personal computers and the incompatibility of software is also a major hurdle.

As described earlier, audio cassettes seem to have more immediate potential. Bates (1982) comments on this, writing:

> I therefore see audio cassettes integrated with correspondence material as a major area for development in distance education: they are cheap, easy to control and make, convenient for students, and above all educationally effective.

One of the major advantages which a combination of audio tapes and textual materials has is that the material can be made more personal. The use of colloquialisms and touches of humour seem less forced on tape than in written form. The material can take on the character of a 'guided conversation' which Holmberg (1981) sees as a very important feature of good distance education material. It is also possible on a tape to help with the pronunciation of mathematical terms. Many distance students who have studied entirely from the written word come to vacation courses reluctant to verbalize their mathematics. They are unsure of the pronunciation of such words as Poisson, Hermitian or even matrices. They are also unsure of the verbalization of symbolic statements such as $sin^2x\ dx$.

Distance education is a very worthwhile and challenging form of out-of-school mathematics education. The rewards are different, but no less real than those of teaching face-to-face.

References

AITKEN, L. R. Jr. 1972. Language Factors in Learning Mathematics. *Review of Educational Research* (Washington, D.C.), Vol. 42, No. 3, pp. 359-85.

AUSUBEL, D. P. 1963. *The Psychology of Meaningful Verbal Learning*. New York, Grune & Stratton.

BATES, T. 1982. Trends in the Use of Audio-visual Media in Distance Education Systems. In: J. S. Daniel, M. A. Stroud and J. R. Thompson (eds.), *Learning at a Distance; A World Perspective*, paper 2. Athabasca (Alberta), Athabasca University/International Council for Distance Education (ICDE).

BRUNNER, R. B. 1976. Reading Mathematical Exposition. *Educational Research*, (Abingdon, United Kingdom), Vol. 18, No. 3, pp. 208-13.

HARRISON, G. 1980. Correspondence Tuition in Mathematics. *Teaching at a Distance*, (Milton Keynes, The Open University), Vol. 17, pp. 30-8.

HATER, M. A.; KANE, R. B. 1975. The Cloze Procedure as a Measure of Mathematical English. *Journal for Research in Mathematics Education* (Reston, Va.), March, pp. 121-7.

HOLMBERG, B. 1981. *Status and Trends of Distance Education*. London, Kogan Page.

OWENS, J. M. R. 1985. *Campus Beyond the Walls*. Palmerston North (New Zealand), Dunmore Press.

SHUARD, H.; ROTHERY, A. (eds.). 1984. *Children's Reading Mathematics*. London, John Murray.

SKEMP, R. R. 1979. *Intelligence, Learning and Action*. Chichester/New York, Wiley.

TATE, O. 1981. *Distance Education; A Case Study of the New Zealand Correspondence School*. Wellington, Department of Education.

WATKIN, A. E. 1979. The Symbols and Grammatical Structures of Mathematical English and the Reading Comprehension of College Students. *Journal for Research in Mathematics Education* (Reston, Va.), May, pp. 216-18.

WILLIAMS, M.; WILLIAMS, J. 1985. *The Evolution and Function of a Student-operated Support Network for Distance Students*. Palmerston North (New Zealand), Massey University Extramural Students Society.

WYNNE-WILSON, W. S. 1978. Examinations and Assessment. In: G. T. Wain (ed.), *Mathematical Education*, Chap. 13. New York, Van Nostrand Reinhold.

Part III

Other sources

7. Education of talented children in mathematics in Hungary

Ferenc Genzwein

The concept of 'talent' and of a talented man are vague, both among those who are expert and in the opinion of the public. Who can be considered talented? Simply those who have an intellect above the average? Or could and should the concept be given a wider interpretation? In seeking answers there are still issues to be clarified by genetics, psychology and pedagogy, though we no longer try to distinguish people with an excellent practical mind or those with good organizing ability from those who are talented in 'theoretical principles.

We should, no doubt, be ready to accept the close connection between developing one's talent and the capacity for diligence, and for regular and steady learning. Nevertheless, we are still in the habit of looking askance at a pupil who gets 'top marks in every subject'.Why is this so? It would seem that we mix the desirable, possible and feasible with that which is realizable only through great effort. If someone gets top marks in one or two subjects through hard work, but gets similar results easily in others, why should we brand such a child a teacher's pet? Why do we not consider the child an example of somone who successfully fulfils the given tasks? Schools should not accept low standards; nor should they confront their talented pupils with unrealistic requirements, either. In the prevailing fog which surrounds the theory and practice of pedagogy, we meet more and more comparisons which are misleading as well as the practice which subsists on them.

One such is the notorious idea of a 'goose'. It goes this way: horses run gracefully, eagles fly marvellously, fish swim wonderfully and larks sing beautifully. They all excel in something. But geese (those who get top marks in all subjects) only shuffle on the ground; they are hardly able to fly, their cries are unpleasant and they are poor swimmers compared to fish. Geese, in fact, cannot do anything. The analogy, however, is false. It would be closer to the truth to argue that geese are equivalent to horses, eagles, larks and fish in that they, too, can do something which none of the others are capable of doing. That is why geese are not average horses, eagles, fish or larks, simply geese. By saying this, I do not want to raise mediocrity to the level of a model to emulate. I wish only to protest against false comparisons. It is the case that, by using a peculiar kind of logic, things that are different are discussed as if they were the same. Thus the different abilities of horses, eagles, larks and fish are lumped together and compared to the abilities of geese.

Many-sidedness is not an enemy of talent. The world-famous, many-sided Jesse Owens, four times champion at the Berlin Olympic Games, took the high jump at 1.83 metres, the long jump with 7.000 metres and he ran the 100 metre flat-race for men in 10.85 seconds. At the age of 12, he set an American long-jump record of 7.61 metres. László Babai, the prominent mathematician won a 'Who is Master in What' competition at the age of 17; he also won, several times, complex school competitions which included tests of nearly all the subjects taught.

In practice, we are often in trouble by using the words 'talent' and 'genius' as synonyms, though it is evident that the number of geniuses is much smaller than the number of the talented or that of men of 'average talent'. It is a fact that there have been and there are men of genius. Gauss at the age of 3 could solve difficult arithmetical problems. Ampère, too, could count perfectly when 4 years of age. Canova was an apprentice in confectionery when his genius in sculpture came to light: he formed such a perfect lion out of butter that he attracted the attention and obtained the patronage of a Venetian senator. Mihály Munkácsy showed his brilliance as a painter when he was painting boxes. So the joiner's apprentice became a world-famous artist. László Lovász wrote well-known scientific mathematical studies when still a secondary school pupil. As a university student, he wrote his dissertation for candidacy. He was barely 30 when he became a corresponding member of the Hungarian Academy of Sciences. Lypot Fejer became a professor at the University of Budapest in 1911, when he was 31. When 30, he formulated a fundamental thesis on the theory of the Fourier series and so started a new trend for modern analysis. Rossini was a lazy pupil, so his father bound him as a blacksmith's apprentice. Davy, one of the pioneers in electricity, was another who did not want to learn. Liebig, the famous chemist, failed his examination for a doctorate. Darwin, too, was lazy at school. His father took him away and bound him as an apprentice to a phlebotomist, a cupper and to a dentist who got rid of him because he failed in his duties. In the event, he became one of the greatest natural scientists ever without having had any regular scientific training. We could go on listing similar examples. Gauss, Einstein, Liszt, Munkácsy and many others became what they were not at school. We know of other examples in cultural history where childhood revealed no brilliance and genius developed in adulthood only. Tintoretto was 30 when his works appeared, and Titian was 40 when his brilliance came to light. He was 99 when his masterpiece was composed.

Schools should draw no conclusion from the above, least of all that they should or could educate Gausses or Darwins. Nor should they think that the way to talent is to be found in failure, nor that what schools cannot do, will be made up for by the talented individual in any case.

In what follows, I do not wish to discuss genius. A genius is a rare exception. He or she can be lucky or unhappy as an individual, and can bring blessing or disaster to a society. The genius is always a special phenomenon. I wish now to discuss the typical talented individuals, whose potential for creation is, or can be developed to a greater extent than that of the average; those who are able to collect their thoughts and actions and extend them to some or to many fields of universal life better than the majority of poeple. Talent, so interpreted, lies hidden in a significant proportion of children. According to the favourable or the unfavourable influences of circumstances (usually the circumstances established by institutional education), it manifests itself, or is latent and maybe wastes away, since an adolescent is still on the way to consolidation. That is why the adjective 'talented' in terms of children and adolescents does not mean a level of preparedness, but a capacity for development which is above the average. It justifies our thinking over and over again the theoretical and the practical tasks we give the talented to do, as the aim of education is at all times the promotion and hastening of development.

Detecting the talented

It goes without saying that talent ought to be recognized at an early stage and that an environment to foster its development should be provided. But to say so gets neither educational policy nor practical pedagogy very far. The statement implies that kindergarten, primary and secondary-level teachers, all practising pedagogues, even university professors, should have the conditions necessary for nurturing the talented. However, we are well aware that the educational climate is rarely favourable to their doing so.

Schools, unfortunately, have never considered (and even now do not take into account) the different inclinations and gifts of different children. Instead, they confront all children with the same tasks. From the beginning of this century it has been educational practice to try to make all children absorb a curriculum of determined quantity and quality at a well-defined pace. So, schools function in a very determined 'order' and the personalities of children can only come to light according to this order. It would hardly be an exaggeration to say that the uniform curriculum planned centrally suits only some of the children to be taught. Instead of children, stress is laid on the curriculum itself, the schoolbooks and other teaching aids.

Another impediment is large classes. These make the work of teachers more difficult in general and especially influence the education of the talented. Different abilities call for different treatment for each individual, the designation of special tasks and the transmission of work most suited to the person. All teachers know how troublesome a talented child can be in class if he or she is not allowed to develop his or her abilities beyond the normal requirements.

One of the greatest problems in childhood and in adolescence is to regonize the trends and the extent of talent. Even if we can correctly determine the inclinations of a child at some given age, it is not certain that the final result will be a good one. Development is hard to predict, and it is as easy to require too much as it is to require too little. There is always the danger that if a child who is considered talented makes slower progress than expected or than those who are set the same work, we are apt to blight that child's self-esteem.

The intelligence quotient

Many think that the intelligence quotient (IQ) of children should be tested more often before and during schooling, because it characterizes their talent by a single number. At present, there are more than 200 intelligence tests available. Indeed, an actual test industry has come into being, especially in the United States. By means of tests, comprehension and command of languages, memory, the ability to draw logical conclusions, etc., can all be tested. The child's performance is scored, analysed mathematically and the final result is expressed by a quotient, the so called IQ.

One of the available tests, maybe the one most generally used world-wide, is the Wechsler test, the first Hungarian version of which was made at the beginning of the 1970s for adults. By this test, it is possible to examine knowledge, counting ability, arranging shapes in a logical pattern, etc. According to the results of this test, the IQ value of one half of all mankind is between 90 and 110; a score of 130 or more denotes extraordinary intelligence, while those below 70 are considered mentally defective. In Hungary about 2.15 per cent are to be found in each of these extremes.

According to Wechsler, intelligence is a combination of factors which enables people to behave purposefully. It is the way in which the individual factors combine which influences the manner in which intelligence manifests itself. The IQ is an indicator, but it doesn't tell the whole story about a talented person's cababilities or about a talented child's potential.

There have been and there still are those who advocate a special school for children with a high IQ. Such experiments were tried in the United States, Germany, Switzerland and other countries at the beginning of the century, but pedagogy has judged them to be disadvantageous. However, revision of judgement continues. The value of intelligence tests is doubted by many on the score of objectivity in measurement; even experts are misled by the way the results of tests are usually expressed. Their seeming accuracy carries the imprimatur of mathematical exactness. Our instinctive respect for a fetish of numbers presumes a precision which no test maker would claim. As Peter Medawar, a Nobel Prize winning biologist, put it: an intelligence quotient consisting of one number makes us believe that we know what intelligence is and that we can even measure it.

According to Endre Czeizel (1981):

> On the basis of a more absorbed study of the results of intelligence tests, Spearman separated, in 1904, 'g' (general) and 's' (special) factors. Naming of factor g comes from its being that common ability which each intelligence test measures. Later on it became usual to call this a general intelligence (cleverness), as, according to our present knowledge, this is

> the ability needed for all mental and most physical activities. When classifying general mental faculties, the following pairs of differences are distinguished: intellectual-instinctive, productive-reproductive, abstract-practical.
>
> Factor s is test specified and means the different special mental abilities measured by different tests. According to the factor analysis made by Thurstone, these are seven primarily mental abilities. They are as follows: orientation, speed of perception, senses for languages and counting, the quickness in finding words, rememberance and logic. According to tests conducted with twins, the quality of different primarily mental abilities shows significant differences within the individuals, too . . . Unfortunately, talent spottings conducted so far and comprising rather a limited spectrum of characters, could not disclose exceptional abilities in case of each child. In accordance with the studies of Ogilvie (1973) and Vernon and his colleagues (1977) which closed with unanimous results, about 2.5% of schoolchildren have average intelligence, and another 2.5 of them have a specific mental ability that can be considered exceptionally good . . . After all these, a sharp discussion has developed between Spearman and Thurstone and, respectively, their followers. As to Spearman, factor g is the determining, while Thurstone stresses the importance of primarily mental abilities in developing talent. Lately, however, these two trends have become quite reconciled.

The main problem posed by pupils selected with tests was that quite a lot of them failed to come up to expectations. Their mental ages as revealed by testing did not develop in parallel to their chronological ages. There seem to be different reasons for this finding. In some cases, a lack of dilligence was noted, though will-power is an important element in selection. It was noticed, too, that in many cases the fact of being selected exerted a harmful effect upon subsequent development. It is important to realize that, when children are made to solve problems or to translate texts in a foreign language, the results are not only an indication of bent; they are also affected by other characteristics such as diligence, determination and a favourable environment. So equality of performance can be achieved by children who are talented in different ways. The presence of special gifts can only be determined if those parts of performance due to 'other factors' can be separated from the total performance.

Other doubts have arisen in connection with the test fetish. According to Guilford, general intelligence, measured by tests and creativity, cannot be identified. He says that test questions have only one right answer, so they value convergent ways of thinking, whereas creative people are characterized by divergent ways of thinking, originality and ingenuity, and not by socially accepted views. Other experts say there is a correlation betweeen an IQ exceeding 120 and creativity, but its coefficient is very small. We teachers prefer intelligence to creativity, which is not yet fully understood. Creative pupils, who may follow their own ideas and behave in a non-conformist way, satisfy teachers less.

Only a small part of talent can be uncovered by an IQ test, and we would do well to remember that the full value of an individual is not determined by intelligence alone. Ethical attitudes are as important as inherent ability and subsequent performance. Experience shows that if someone knows what is good, that person is not necessarily doing it. That is why it is an important pedagogical requirement to exploit each subject, so far as it is possible to do so, for its implications.

We tend to distrust people whose only concern is for performance and results. Our ideal person is characterized by totality: that way of thinking which values others for what they are. We can conclude from this that while intelligence tests have a useful part to play in pedagogical practice, they cannot be accepted in themselves as infallible methods of measuring talent. This conclusion is important, because some practising teachers and other experts have an uncritical trust of intelligence tests, while others regard them disdainfully. The former indulge in a fetish of tests, supposing them to give exact results, while we can only conclude a certain probability from them. So, in seeking to nurture talent, it would be a wrong solution to select children by giving them an intelligence test and put them into a special school.

Possibilities and tasks to be done in the field of talent-improvement

The uncertainty which surrounds nurturing talent casts doubt on the possibility of finding a satisfactory solution in educational policy and school practice. It would seem that methods based on the means we have for detecting talent and for its mass development are hardly available. Science will either solve this problem in the future or not. We are aware, however, that experts in different countries have not yet abandoned the search for a solution to this very difficult problem.

My own conviction is that the job of determining and developing faculties, abilities and talents should be primarily the task of schools. Other institutions should take initiatives in this endeavour, but schools can undertake the task only if they become more open than they are now, able to perform their duties in a more professional way.

Clearly, a single rigid 'central arrangement' cannot lead us to our goal. The main task seems to be to operate our schools on the basis of an *intensification programme*, in the best sense of the word, using all those up-to-date curricular settlements and pedagogical processes to which the talented will respond. They include the different experiments conducted in this country, instruction in the mother-language based on research into communication, the experimental school in Szentlőrinc, the teaching of mathematics in primary schools, the time-honoured formation of special classes with sections, the full potential of faculties, etc. We can hope for greater activity and productivity in schools than today if we draw all the lessons to be learnt from the above and further elaborate their methods. Only then can we expect the individual work of pupils during class to increase, and through this, an increase in creativity, too. In this way, we can hope that individual perception and experiment will play a greater role, so that pupils can be more active in acquiring knowledge. It is not a child's potential abilities which is the most important factor: it is the ways and the frequency of their employment. Talent develops through activity.

Those who are interested in the development of the inner life in our schools know that, in spite of the many tasks to be done, there is ambitious work going on to transform schools into creative workshops. We should keep in mind that a person who is able to produce something outstanding and valuable, has always had to discipline himself or herself to a hard work, but that this discipline has brought pleasure. Indeed, such a person will explain the goals and reasons for life only in this way. Gorki put it like this : *talent is work*. The moral for teachers is that they should avoid becoming stuck-in-the-muds. They should seeek renewal in stimulating dissatisfaction with themselves and the desire for self-development. So we need educated teachers with receptive minds, since the most important factor in detecting and developing talented children is a good teacher. Many have proved and still prove that there have always been teachers who have educated exceptional children *en masse*, with the help of their own perceptive resources. These teachers are able to create a sense of success in their pupils and enable them to envisage an aim most suited to their own personality. They know that the aim of school is not to educate miraculous children, since to do so would be unreal and its result of questionable value.

In selecting and educating the talented, even tradition can help us. Sucessful, already beaten tracks can be adopted from the past. In Hungary, as far back as 1894, the Eötvös Competition was organized and held every autumn. This enabled those who left their secondary schools in a particular year to demonstrate their talent in open competiton. The two best performances were rewarded and published in a periodical of the János Bolyai Mathematical Society. Many mathematicians who later became famous attained their first scientific success in this way. These competitions also tested talent. Not all talented pupils could enter them, but it provided proof that those who won were talented. The competition was regarded as fair because it build upon a rather moderate knowledge of mathematics and mainly examined ways of thinking, ingenuity and the ability to use these attributes properly. It is important to point out that participants prepared themselves for years in order to be able to take part successfully in this famous competition. They were helped by the *Secondary School Mathematical Papers* (a periodical) established in 1894, and by the 'Mathematical Competition Theses' issued in 1929. These contained the problems and solutions of the first thirty-two competitions and also published valuable notes. From 1949 onwards, this competition was carried on under the name 'József Kürschák Mathematical Competition for Pupils'.

There is no lower age limit for competitors and this is important. Younger pupils have won many times. The publication 'Mathematical Competition Theses' has been brought out several times since then, and it offers useful reading for interested pupils and their teachers. Learning from this experience, the 'Dániel Arany Competition for Pupils' with its several rounds, and the 'National General Competition Between Secondary Schools' have been established. They have attracted masses of pupils. The number of those competing had grown to 3,000 by 1962 and to 10,000 by 1985. Furthermore, the International Mathematical Olympiad for Students has been organized for many years. Hungarian participants always meet with great success during this competition. The mathematical competition called 'Miklós Schweitzer Mathematical Memorial Contest' is of the highest level. It is held mainly for university students, though younger pupils can also enter the competition and have frequently met with success.

The above-mentioned *Secondary School Mathematical Papers* gives a great boost to pupils. This is a popular periodical amongst thousands of pupils and hundreds of teachers. The scoring system established in this paper is a means to developing the pupil's abilities and their ways of learning. It also includes valuable articles written by academicians.

As far back as 1950, Debrecen Tibor Szele brought into existence a successful mode of educating talented pupils. He provided them with the possibility of learning at a higher level than in school circles in the course of what he called his 'mathematical afternoons'. At present, there is a competition form for primary school pupils, too, and we have set up a wider range of possibilities for learning outside schools as well.

In Hungary, the system for educating the talented has come into being as a consequence of efforts made by enthusiastic people. The system covers even more possibilities that we actually make use of. The main problem is that the news of the various possibilities does not reach every school and there are many blank spots even now. The extent to which pupils can profit from the possibilities depends on the needs, the schools and the enthusiasm of teachers. The system is one which could and should be developed.

Reference

CZEIZEL, E. 1981. Is Exceptional Mental Ability a Disadvantage? *Kritika* (Budapest). (Published in Hungarian.)

8. Mathematics in literacy classes

Raymond Zepp

Introduction

The term 'literacy' originally meant the ability to read and write. In recent years, however, the term has grown to include the ability to perform arithemtical calculations: addition, substraction, multiplication and perhaps division. The mathematical side of literacy is often called 'numeracy', and this word will be used in the discussions which follow.

Large numbers of adults around the world are both illiterate and 'innumerate'. These adults do not have the time to attend school full-time, and, even if they could, there are few schools designed specifically for adults. Moreover, it is widely recognized that the learning process for adults is often different from that for children, and that formal schooling is not always beneficial for adults. Rather, the emphasis has been placed on various kinds of 'non-formal' education, by means of which adults can learn to read, write and compute in their spare time, at their own speed and in ways suited to their specific educational needs.

In the past, the numeracy aspect of literacy training programmes was given, at most, lip service, or was often treated as an appendage or afterthought to the reading classes. Once the adults could read, they could then use their newly acquired skills in reading materials related to arithmetic. This 'post-literacy' training had as its primary objective the consolidation of the reading skills; the actual utility of numeracy, while recognized, was treated as secondary to the reinforcement of reading skills.

The separation and isolation of numeracy skills from reading skills is still widespread in literacy courses. This separation is perhaps due to the specialization of those who design the projects. Literacy is traditionally handled by linguists, not mathematicians. If a numeracy component is to be added, a mathematician must be called in. It has only been in recent years that the interrelations of numerical training and the teaching of writing have been recognized; research into these interrelations is as yet in its infancy.

On the one hand, mathematical skills can be used to enhance the teaching of reading and writing. In many cases, it may make sense for adults to learn arithmetic as a prelude to the study of reading and writing, that is, as 'pre-literacy' training. This idea will be developed in more detail below. On the other hand, educators are learning that the adage 'mathematics is a universal language' may need to be re-examined. There are linguistic hurdles to overcome in learning arithmetic and each language has its specific set of hurdles. These problems must be considered in teaching numeracy to adults in various languages. Such problems are also discussed below.

In recognition of the connections between numeracy and literacy, the remainder of this article will attempt to discuss the role of numeracy training in literacy courses.

Objectives of numeracy training

In discusssing the role of numeracy training in literacy courses, one must not only pinpoint the objectives of numeracy training *per se*, but also consider the extent to which these objectives are in harmony with the goals of literacy training. The objectives can be grouped into three categories: functional objectives, educational objectives and social objectives.

Functional objectives

These are the direct objectives which are the most obvious to the participants. Most common among these objectives are reading and writing sums of money, reading scales in measuring or weighing, and calculating prices in market situations. The need for acquiring these skills is acutely felt by most innumerate farmers around the world. The following scenario is typical of the situation in many developing countries:

The residents of a small isolated village in West Africa sell their coffee to the coffee company, whose representatives pay weekly visits to the village. The company representatives weigh the coffee beans themselves, using their own balances, which the villagers cannot read. The villagers must accept the world of the representatives as to the weight of the coffee. Then, although the price per kilo is fixed, the representatives multiply the weight by the unit price to obtain the amount to be paid the farmer, who must again accept the representatives' word, since he cannot calculate the price by himself. The villagers correctly perceive that they are at the mercy of unscrupulous traders. It was not surprising, therefore, that when a literacy campaign was launched in the village, the villagers stated emphatically that they were far more interested in numeracy than literacy.

Indeed, whenever numeracy training is offered to illiterate adults, the response is almost always overwhelming. Learning to calculate prices is a goal which adults readily perceive as affecting them directly and tangibly.

Educational objectives

For many adults who have never been to school, literacy/numeracy classes may be the first exposure to a 'schooling' experience. 'Learning to learn' is very important to the adult who may subsequently wish to acquire other skills. Numeracy training can be used to achieve some of the fundamental skills needed for further education. In particular, numeracy training can be used to achieve objectives related to 'pre-literacy'.

Beginning to learn to read and write can be a trying experience for many adults. The learner is at once faced with a host of concepts and skills with which he may be unfamiliar, but which he is expected to master almost immediately. Examples are motor skills (writing), interpretation of symbols, memorization of names (the names of letters), etc. Numeracy training, as pre-literacy experience, is an attempt to ease the learner gradually into the use of these skills. Basic arithemtic may be easier to master than reading for several reasons:

There are only ten numerals (0,1, . . . 9) as opposed to twenty-five to thirty letters.

The learner, in his native language, already has names for the numerals, whereas he must learn new nonsense symbols as names for the letters. For example, the learner who speaks only French, say, knows what '*un*' means, and he can readily connect that name, along with the concept, to the symbol '1'. On the other hand, he may not be used to giving names to sounds. Thus he must not only formulate the concept of the sound represented by the symbol 'Y', for example, but he must also memorize a new name for that concept, a nonsense name in French ('*i grec*') which has little connection with the sound.

Teaching methods can be much more visual and concrete for number symbols than for letters. For example, by counting a pile of five stones, the learner can see and touch the number 5, whereas the letters represent strictly aural concepts.

An equation, such as '4 + 5 = 9', represents a simplified kind of 'sentence'. Symbols are juxtaposed in order to form a complete thought and this thought can be read simply by pronouncing the symbols from left to right. While equations illustrate fundamental ideas of sentence structure, the reading of such sentences is, although easier, fundamentally different from the reading of sentences of words. Letters must first pass through the intermediate stage of words before they can be understood as a sentence. In order to read the sentence, 'He is bad', one must first put the sounds H and e together to form the recognizable word 'He'. Reading equations omits this step: one simply pronounces the symbols in order. This would be equivalent to reading the above sentence simply as 'aytch, ee; eye, ess; bee, ay, dee'.

Thus, numeracy training may have such educational objectives as attaining the motor skills of writing, interpretation of written symbols, connecting symbols into ideas similar to sentences and abstract thinking. Such objectives, while difficult to evaluate, can be taught via the vehicle of numeracy training, which has a direct motivational appeal. The innumerate villager may be unaware of these objectives, as he attends numeracy sessions simply because he does not want to be cheated at the market. But the designers and teachers of numeracy courses should be aware that they are also shaping the minds of the participants for further educational experiences.

Social objectives

One of the major obstacles to many community development projects is the difficulty in motivating people to feel the need for development. Community solidarity is far from evident in many Third World villages. One of the goals of literacy/numeracy projects is to bring members of the community together. To be sure, each participant has his own individual goals, but the organization of the classes requires group participation. Participants are encouraged to help one another in learning the material. In this way, an *esprit de corps* is built up; a desire for the class to move forward as a whole. The ability to work in groups is useful in all community development projects, and so the designers and teachers of numeracy/literacy projects should bear this goal in mind at all times.

Methods

Adult numeracy/literacy training is usually classified under the heading of 'non-formal' education. In designing such a programme, one must ask the question 'How informal should "non-formal" be?' On the one hand, illiterate adults cannot be expected to find the time and money for full-scale formal schooling. On the other hand, self-study or 'distance teaching' materials can be virtually ruled out because the participants cannot read the instructions. In fact, an adult who has had no schooling will need all the individual attention he can get from some authoritative 'teacher'. In the light of the social objective cited above, it would seem desirable for the participants to meet together in something resembling a classroom situation.

Although resembling the classroom in some respects, most numeracy programmes will differ from formal educational settings in several ways.

First, the 'teacher' is not seen as a figure of authority who dispenses knowledge and keeps strict discipline; rather, he should be seen as a helper. In fact, there are often several helpers: numerate friends and relatives often volunteer to assist the innumerate. Thus the terms 'student' and 'teacher' are rarely applied to non-formal education; they are replaced by such terms as 'participants' and 'monitors'.

The non-authoritarian role of the monitor is especially important in societies where status is accorded to age groups. Village elders may resent being placed in positions subordinate to younger educated teachers and the monitor must, therefore, keep a low profile. The focus is on the learner; the monitors are seen as simply one of many aids to the learning process.

Second, if there is no single teacher lecturing to a passive audience, then it follows that there need be no formal classroom. Non-formal learning often takes place out-of-doors, on porches, in private homes, etc. However, the local school seems to be a favourite meeting place for logistical reasons:

A classroom is one of the few places where large numbers of people can be seated comfortably at tables or desks necessary for writing.

Villagers who work during the day may prefer to meet at night, when lighting is a prime consideration (a schoolroom may be one place which is likely to be lighted).

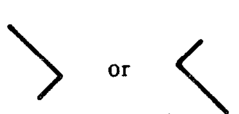
Other educational materials such as chalk and blackboards may be available in schools. Further to this final point, small individualized slates are perhaps more conducive to non-formal learning than large blackboards, for in using the small boards, each participant can do all the writing he wants, on his own.

Third, the organization of numeracy courses must be more flexible than that of schools. The adults participate of their own free will, they know what they want to learn and they are not simply seeking pieces of paper in the form of diplomas, titles or certificates. There can be no attempt, therefore, to force curricula or other rules upon the participants. It makes little sense to penalize volunteer participants for tardiness, absenteeism, etc. In a real sense, the course belongs to the participants and it is they who must ultimately determine the organization, scheduling and curriculum.

Curriculum

In discussing the topics covered in numeracy courses, it is useful to include in the discussion the methods by which the topics are treated. The methods will naturally vary, depending on the materials available, cultural preferences and specific objectives.

Writing numerals. The writing of numerals poses several difficulties for illiterate adults. The educated person may deem these difficulties trivial, but for the unschooled, writing presupposes a complex mind-set which is difficult to construct overnight. For example, literate people naturally think in verticals and horizontals, but illiterates may not have acquired such concepts. They will attempt to write numerals at any angle, perhaps even upside-down. When asked simply to copy say, the numeral 7, an illiterate may well write



Such a person may be able to distinguish isomorphic copies without grasping that a translation without rotation is called for.

It usually takes hours of practice for many adults to learn to write the numerals from 0 to 9. Perhaps the most important quality of the monitor is patience. Several methods can be used to enhance the learning process. Before the actual writing takes place, participants may be asked to visualize the number in the air and to trace it with their fingers.

The formulation of vertical and horizontal concepts can be aided by visual terminology. For example, a project in Haitian creole was successsful in using the terms 'standing up line' and 'lying down line' (*liy ki kampe, liy* *ki koushe*). But most important is simply practice, with as much individual attention as possible. The mere observation of a teacher's performance or that of a classmate at the blackboard is not sufficient for illiterate adults: they must do it by themselves.

The reading of two-digit numbers. First of all, it is not always an easy task to get adults to read numbers from left to right. There is sometimes confusion between, say, 21 and 12. But far more disturbing problems arise when the language of instruction is one where counting is not done in base ten. This problem will be discussed later, under 'Linguistic considerations'.

In order to teach the concept of place value, it is best to proceed as concretely as possible. Many of the adults have not reached the formal stage of reasoning in some domains and the worst mistake a monitor can make is to assume that abstract reasoning can be followed. Thus, counting can be done with stones, nuts, cigarettes, etc. placed in piles of ten. Perhaps each pile of ten can be replaced with some other marker, for example, ten cigarettes, can be replaced by a cigarette pack. Then the number 57, say, can be seen as 5 packs and 7 cigarettes. If the proper coins/bills are available, money can be used to convey the concepts of hundreds, tens and units.

Operations. The operations +, -, and x are always included in numeracy courses, while division is sometimes omitted due either to its relative complexity or to shortage of time. Again, the instruction must be as concrete as possible. An abstraction such as '3 + 4 = 7' should be accompanied, whenever possible, by adding of real objects. Multiplication, especially, should be done in real life situations. For example, a market scene can be simulated whereby one participant buys, say, five objects from a second participant at $6 per object.

The choice of algorithm and method of algorithmic instruction may vary considerably, since algorithms can be complex and tedious. In many cases, however, computational aids such as electronic calculators or multiplication tables obviate the need for complex algorithms. Such aids will be discussed below, under 'Combatting dropout'. The mastery of complex algorithms and the memorization of multiplication tables require a great deal of the participants' time and can lead to frustration and dropout. Thus the need to simplify or to eliminate complex algorithms is one of the most important and challenging features of adult numeracy programmes.

Measurement. Adults all over the world need to be able to measure things: linear distances, weights, volumes and areas. In many cultures, however, the units of meaurement are neither precise nor standardized. In the marketplace, rice may be sold by the 'basket', but each basket is different. Many adults have difficulty therefore with the new concept of standardized measurement. It may be necessary, for example, for each participant to measure a given line segment with his own metre stick in order to verify that all metre sticks yield the same measurement.

The concept of conservation (of volume, etc.) requires, according to Piagetian theory, a certain mental development on the road to abstract reasoning. Many adults in remote areas may not have reached the formal stage of reasoning in the area of conservation and may need much coaching to arrive at the abstract concept of 'litre', 'metre', etc. For example, it may be useful for a participant to cut a piece of string one metre in length, then to repeat the process, and finally to compare the two pieces of string to verify that they are equal in length.

Units of measure vary widely; an ethnic group may use a traditional set of units completely different from the official set used by their government. The choice of units used in numeracy classes can be a sensitive issue for political reasons.

However, if the participants are to feel that the numeracy course is theirs, they will usually be permitted to choose the units of measurement.

Many cultures do not measure surface area in square linear units such as square metres or hectares. The measurement of an irregularly shaped field can prove very difficult indeed. Many cultures measure fields by perimeter and may not be aware that more crops can be planted in a square than in a rectangle of the same perimeter. Other cultures use methods of measuring areas which are even stranger to Western ideas, for example, estimations of the length of planting time, or the amount of grain it will yield. Some clans of Mossi in Burkina, who build circular houses, measure area in circular units. This method is convenient, because an area can be expressed by the radius of a circle.

Related to the impreciseness of measurement in many cultures is the absence of the concept of fraction. Even the term for 'one-half' often means only 'a part', which could also represent one-third or one-fourth. The number $3^1/_2$ may have not direct translation, other than 'three and a part', a translation which applies equally to $3^1/_3$. For this reason, fractions are often omitted from numeracy classes. The notion of decimals can be introduced by defining a number such as 2.16 as '2 metres and 16 centimeters'. This method works well as long as all numbers with decimals points have two figures after the decimal point, such as 4.05. But real confusion can arise when a number such as 4.5 is introduced: it will almost certainly be taken to mean 4 metres and 5 centimetres. In short courses, therefore, it may be deemed expedient, although perhaps intellectually dishonest, to treat all decimal numbers as having two figures to the right of the point.

In any case, the numeracy class should use the same *instruments* of measurement which the participants will use in daily life. If, for example, the local co-operative weighs cocoa using a hanging balance, then similar balances should be used in the numeracy classes for instruction in weighing. An educated person may consider it trivial to transfer from one kind of measuring device to another, but an unschooled beginner may find it quite difficult.

Currency. Under normal circumstances, the participants will have enrolled in the numeracy classes primarily because they wish to compute monetary sums. It makes sense, therefore, to introduce units of currency into the numeracy material as soon as possible. However, there may be complexities or linguistic subtleties which render it desirable to delay the introduction of currency until later in the course. Such difficulties will be discussed below under the heading 'Linguistic considerations'.

To render the classes as true-to-life as possible, real money is often used in calculations. Simulation of market activities can be a motivational tool, especially if real coins and bills are used. In countries where more than one currency is used, conversion from one currency to another can be a useful exercise in multiplication and division.

Currencies based on decimal systems can serve as a good introduction to decimal notation. One dollar and fifty cents can be written $1.50 and then the same notation can be generalized to any objects which can be broken into one hundred equal parts.

Geometry. Formal Euclidean geometry is not often considered in the curriculum of numeracy courses. There are, however, certain basic geometrical concepts which are often introduced, perhaps simply to facilitate the reading and writing of symbols. For example, the concepts of parallelism and perpendicularity as often absent from certain cultures. In order to write certain symbols such as L, 4, T, +, etc., it may be useful for the participant to be able to recognize and to draw perpendicular lines. Similarly, while most cultures have some concept of roundness, the more precise notion of 'circle' is often absent, and is a useful idea for the participants to acquire.

One controversial topic related to the introduction of new geometrical concepts into a language is the problem of how to name the new concept. Should the Western word e.g. 'circle' (in English) be used, or should a descriptive phrase be constructed from words in the local language? For example, Haitians usually use the creole world 'wonn' to indicate anything round. It was necessary, in developing a numeracy course for Haitians, to decide whether the concept of circle should be denoted *sek* (French, *cercle*), or by some descriptive phrase such as *wonn egzak* (exact round). The controversy in such situations usually centres on the preferability of simple but foreign phrases as opposed to more complex but linguistically purer phrases.

Combating dropout

In literacy programmes around the world, perhaps the greatest single problem has been that of dropout. Dropout rates of well over 50 per cent are quite common in even the most conscientiously thought-out literacy projects. Adults apparently have a keen desire to learn to read, but find the process much more tedious and time consuming than they had imagined. It is impossible to learn to read in a month or two, and the fruits of one's endeavours are not immediately apparent. After the initial motivation with its high expectations, frustration and disappointment with slow progress are experienced, leading to dropout.

In some respects, numeracy training is less vulnerable to dropout than literacy training. For the participants, visible results are forthcoming. After a lesson on measurement, for example, the participant can immediately go into the real world and measure things. Feedback is immediate; the participant can see that he is making progress.

In other ways, however, numeracy training suffers from the same defects as literacy training. First, the material is cumulative in the sense that each lesson builds on the material of the previous lesson. If a participant misses a lesson due to illness or other obligations, he is totally lost when he returns to later lessons and will probably drop out. Second, participants progress at amazingly different rates. The pace of the classes is likely to bore bright participants or to outdistance the slower ones. Third, the complex algorithms used in performing the operations of addition, subtraction, and especially multiplication and division, take a long time to master. The memorization of multiplication tables can be a boring and seemingly meaningless procedure.

The first defect is inherent in the nature of the material and little can be done to overcome it. One partial solution is to present the same lesson twice. Participants may attend one session or the other (or both, for extra practice) without missing any new material. Alternatively, participants who have learned the material may be asked to help others who have been absent.

The second defect can be corrected partially by a method of multiple sessions just mentioned. Another method is to use the faster students as helpers for the slower ones. This method has the added advantage of reinforcing the just-learned skills of the faster students, as well as adding to the class group solidarity. In fact, the desired picture of non-formal edcuation is usually not that of a teacher lecturing to a disciplined class, but rather a scene of adults working in groups of 2, 3 or 4 helping each other master the material.

The third defect, that of the length of time required to master the algorithms, is perhaps the most serious. Centuries of experience have shown that the algorithms or their methods of instruction will probably not be simplified a great deal, although methods may be found which are perhaps more amenable to adult learning. Most of the attention today is given to ways of eliminating the algorithms entirely by using computational devices. A step in this direction is the use of written multiplication tables in order to eliminate their rote memorization. But the most promising innovation is the use of electronic calculators to eliminate completely any learning of algorithms for any of the four operations. The use of calculators appears so promising, in fact, that it is discussed in detail below.

Use of pocket calculators

Calculators are becoming more and more popular in developing countries. Prices have dropped and low income groups can now afford a simple machine capable of performing the four operations. It is therefore appropriate to consider their use as an educational device among illiterate adults.

In numeracy courses, the pre-eminent advantage of calculators is that adults need not spend tedious and frustrating hours learning the algorithms for performing the various operations. Calculator use, however, offers additional advantages.

First, electronic devices have a tremendous motivating force in most developing countries. Many rural dwellers are fascinated by technology and gadgetry. The possession of any electronic device and the ability to use it carries high prestige in many villages. Thus the motivational qualities of the calculators may keep participants interested and hence prevent dropout.

Second, in conjunction with the 'pre-numeracy' goals discussed earlier, calculators are excellent educational aids to symbol recognition and manipulation. Immediate reinforcement is provided when the symbol of the key pressed appears instantly on the screen. Morever, in order to perform a computation, one presses the keys in the 'sentence' order of the equation. For example, to compute 4 + 5 is to complete the sentence '4 + 5 =   ', the symbols being entered in the correct left-to-right order

One may object to the use of calculators on theoretical grounds: participants may press buttons mindlessly without any appreciation of the size of their answers. Innumerate adults who have no 'number sense' may, through errors in keypunching, arrive at answers all out of proportion to the type of numbers involved , without being the least bit surprised. For example, if, in attempting to multiply 4 by 5, a participant sees an answer of 2,000 on the screen, he may not realize that anything is amiss.

The objection is a valid one. It must be combatted by trying to instil in the participants some number sense. This can be done by estimation of answers before actual calculator use. In practice, however, the problem rarely arises in real life: in concrete situations, the participant will have a sense of the size of the answer expected. For example, if a farmer sells 4 kilos of rice at 5 pesos per kilo, he will know immediately that the calculator should not give an answer of 2000 pesos. This is yet another reason why numeracy classes should keep as close as possible to concrete examples. Participants may not have a number sense in the abstract, but in concrete matters which touch them directly, they can be exceptionally astute.

The selection of calculators is important for the success of numeracy training. Participants should not be led into confusion by fancy models with many buttons and mysterious symbols. Simplicity should be the key criterion in machine selection, followed closely by durability. The machines may receive rough treatment. Spare batteries may be difficult to obtain and so such features as extra long life batteries and automatic shutoff are important, since participants often forget to switch off the machines. Recently, solar-powered calculators have come onto the market, and these may be useful in remote areas. A final criterion is legibility. Many participants have very poor eyesight and find small characters difficult to read, especially in the poor lighting in many of their environments. Thus, it is advisable to use large machines with large, legible characters.

Linguistic considerations

Although 'mathematics is a universal language', there are many aspects of basic arithmetic which are highly dependent on language. The symbols we use have been invented to correspond to words which are conveniently expressed in our language. When the same symbols are read by persons of other linguistic groups, they may not be so convenient and may be even more confusing than helpful. Special care must be taken to lead participants over these hurdles.

In languages where the numeral system is not in base ten, the reading of numbers above ten is confusing and unnatural. If a participant is used to thinking in twenties instead of tens, he is likely to interpret a number such as thirty-one as 'three twenties and one unit.'

Some languages count in twenties *and* tens *and* units, so that the number 312 may be taken to mean three twenties and ten and two, or seventy-two. For an adult to read numbers written in base ten while he is thinking in base twenty is perhaps the most difficult obstacle to overcome in the numeracy classes. One obvious way of attacking the problem is to count concrete objects by piles of ten. Thus a participant can sort, say, forty-five objects into four piles of ten with five remaining, and hence write 45. In the reverse order, given the figure 45, he can count out four piles with five remaining, and then regroup them in order to restate the number in his own language system. This concrete method works well, but it is quite useless in practice. In the marketplace, the farmer cannot count out forty-five objects in order to interpret the number 45.

Rather, some rote memorization must be applied. For example, the Frenchman who sees the number 80 does not need to count out four groups of twenty to arrive at the word '*quatre-vingt*'. Instead, he simply has memorized that 80 means '*quatre-vingt*' and in prnouncing the word, he is hardly aware that he is counting in base twenty. Similarly, a person counting in base twenty will simply have to memorize that 100 is five twenties and therefore a number such as 345 means 3 times 5 twenties plus 4 tens (or 2 twenties) plus 5'. In short, in order to use base ten notation, one must be taught to think in base ten and to ignore the deeper, base twenty meaning behind the number words.

Problems can also arise when the local language assigns to coins and bills words which do not correspond to the numbers printed on the currency. In the franc zone of Africa, many languages call the smallest unit of money 'one', even though that coin is marked '5F'. Thus a coin with the symbols '100F' is called 'twenty' instead of one hundred. The participants expect the coin to be labelled '20F', and become confused when they see instead '100F'.

It is likely that even the concepts of +, -, x and ÷ are dependent on the language involved. In one language, addition may be treated as 'bringing together', while in another it may be translated 'putting on top of', and in a third there may be different expressions for the addition of things, the addition of animals and the addition of money. One must realize that the symbol '+' represents a highly Westernized abstraction which may not fit in well with other languages or cultures. A second example is the symbol '<', which means simply 'less than' in English. But in many languages, comparison is expressed by some word which means, roughly, 'surpasses'. In such cases, it is possible to say 'five surpasses 4', i.e. '5 > 4', but there is no convenient way of saying '4 < 5' without inverting the sentence order, or using the word for '>'. A third example is the symbol '='. In most languages, this symbol will be translated quite differently in the expressions '3 + 4 = 7' and '7 - 4 = 3'. In the first '=' is usually translated by the equivalent of 'result', 'total amount' or 'reaches up to', while in the second, using subtraction, '=' is equivalent to 'remains', 'left over' or 'down to'. It may therefore be difficult for participants to accept that the same symbol '=' should be used to express the results of both addition and subraction.

In assessing pre-literacy goals, 'sentences' such as '3 + 4 = 7' are introduced to habituate participants in reading sentences from left to right simply by reading the symbols in order. But the translation of the sentence into other languages may involve quite a different word order, in which there is no simple one-to-one correspondance between symbols and words, as there is in the English 'three plus four equals seven'. The translation in another language might be equivalent to 'the result of taking three objects, four objects, putting them together, seven objects.' For such a phrase, a more appropriate sentence order would be '= 3 4 + 7'.

In general, however, it is probably fair to say that the linguistic difficulties associated with the symbols for the operations are minor. Adults usually can adapt to differences in word order and operational concepts as long as the arithmetic is done in the light of real-life situations. The farmer can easily conceptualize that if he has three goats and buys four more goats, he will have seven goats in all, regardless of the language in which he expresses his thoughts. One report of difficulty, however, comes from the Gambia, where participants experienced some confusion over subtractions such as '5 - 2' because the sentence order in the local language was 'take two away from five' (the 2 came first).

Evaluation

The evaluation of numeracy courses is extremely difficult in the light of the various objectives of the course. It is of course easy to calculate the dropout rate; which is a rough index of the popularity of the course. But there are many outside influences which affect the dropout rate: weather, illness, political unrest, etc. Thus it may be grossly unfair to claim that a dropout rate of 40 per cent in one region indicates a more successful programme than a dropout rate of 60 per cent in another region. Moreover, one must ask whether the length of attendance should be taken into account. Is dropping out after one week equivalent to dropping out after six weeks? Has a person who has dropped out after three weeks perhaps learned more than another who has attended all of the classes?

It is also easy to administer a pre-test/post-test to participants to ascertain whether they have mastered the basic skills. One may ask them to perform simple calculations, measure certain objects or calculate prices of market items. These are the 'functional objectives' discussed earlier, which, on a scale of objectives such as Bloom's taxonomy, are the easiest to acquire and the easiest to evaluate.

More difficult to evaluate are the 'educational objectives' related to pre-literacy training. How can one measure whether numeracy training has aided participants in subsequent literacy classes? One might suppose that it would be possible to see whether those who had acquired numeracy skills performed 'better' somehow than a control group in the literacy classes. But the numerate group would consist precisely of those students who showed the perseverance and motivation to complete the numeracy course.

One must rely almost entirely upon subjective evaluations. It is not easy to administer questionnaires to illiterate adults. Instead, one must detect attitudes through interviews. The monitors are those whose fingers are closest to the pulse of the participants, and they can be a source of much evaluational information. For example, one monitor reported that the participants had used the symbol '+' to denote the combination of sounds, for example, '$b + a = ba$'. Such a revelation was an indication that the numeracy training had had a beneficial effect on the literacy training. One must also at least entertain the possibility that the numeracy training might somehow detract from the literacy training. Perhaps some participants become confused because letters do not behave, as the participants expect, like numbers. This is unlikely but possible. It would be difficult to document any such interference statistically.

Finally, the 'social objectives' are the most difficult to evaluate. How can one measure group solidarity among villagers and how can one verify that such solidarity, even if measurable, was attributable to the numeracy classes? Besides, group solidarity is highly dependent on the personalities of those involved; of two groups given exactly the same treatment, participants of one group may get along well, while those of the other will squabble. Moreover, it takes years of working together to develop community spirit. A numeracy course of one month's duration will not of itself cause a community to evolve the spirit of development. But if such courses are run repeatedly, along with other community projects, members of the community will become accustomed to working with one another, especially if they begin to see positive results attributable to the development projects.

Summary

In general, adult numeracy programmes seem to fit conveniently into literacy projects. The two areas share similar goals and one set of skills can be used to complement the other. Although numeracy training is difficult to evaluate, the results of interviews and other subjective assessment seem to indicate that numeracy courses can indeed achieve all three types of objectives: direct functional objectives, literacy objectives and social objectives.

In fact, numeracy courses often seem to be more successful than literacy courses, for several reasons:

1. They last a shorter period of time. An adult can learn to read numbers, measure and compute in a period of under two months if computational devices are used to eliminate complex algorithms; it may take a year to learn to read effectively.

2. Results are immediately visible: an adult who learns to read numbers can use that skill immediately in any market activity. The immediate feedback provides the motivation to keep the participant from dropping out.

3. The prospect of financial gain is highly motivational. A farmer may wish to read books, but if he thinks he will gain financially from numeracy training, he will prefer that to literacy training.

On the other hand, there are many reasons for which numeracy courses can and do fail.

1. As numeracy does not enjoy the high status of literacy in governmental and other funding agencies, numeracy programmes are usually very poorly funded. Teachers are usually volunteers and materials needed to render the instruction more concrete are usually unavailable.

2. Numeracy projects are often launched by foreigners who do not appreciate the cultural and linguistic variables which affect the organization and method of instruction.

3. Projects initiated from outside the community (e.g. government, church, etc.) are often viewed with suspicion and distrust by the participants. They should feel that the programme, along with its decision-making, is theirs, rather than something foisted upon them by outsiders. Xenophobia, irrational or otherwise, should never be underestimated.

From the above, it should be clear that the planners of numeracy programmes must be specialists in more than one field simultaneously. The training of experts in numeracy must provide a solid base in mathematics, linguistics, sociology and development economics. Numeracy projects must be organized by broad-minded, broadly trained experts sensitive to subtle cultural and linguistic factors.

Numeracy training seems to be in great demand in developing countries, among people who are exploited because of their innumeracy. However, the expression of that demand is not always heard or transmitted by those who are exploiting them, and thus numeracy training has always suffered from low priority. If the news of successful projects could be spread more widely, then perhaps numeracy training would experience a greatly enhanced popularity around the world.

9. Mathematics training for work

Rudolf Straesser

What about work?

In writing a paper entitled 'Mathematics training for work', one is obviously faced with the problem of forming an idea about the meaning of 'work'. Sowing corn to sell the crop to a mill, fixing a wheel in an assembly line to earn one's living or reading a paper to prepare a lecture on mathematics education could easily be described as 'work'. But what about cleaning the pot after a meal or attending a debate on health and welfare in parliament? What about scoring the decisive goal in a football match, or shooting a tiger to exchange its fur for food?

The *Advanced Learner's Dictionary of Current English* gives seven explanations under the key-word 'work'. The first two are: 'bodily or mental effort directed towards doing or making something'; and 'occupation; employment; what a person does in order to earn money' (Hornby et al., 1960, p. 1492). If we look at these explanations, we find two fundamental aspects of 'work', which are not necessarily occuring together. Activities (bodily or mental) are called 'work' if they are performed on purpose and 'work' seems to be related to earning one's living sometimes by exchanging goods directly, sometimes by selling goods or one's labour, thus getting money. These different patterns of the organization of work reflect a major difference between industrialized and developing countries. The existence of an economy which is based on the exchange of money for goods or labour nowadays characterizes industrialized, developed countries, whereas a direct exchange of goods and labour play an important role in rural, developing parts of a country. Obviously this distinction has consequences for training for work. The first explanation also hints at an important line of division of work: the separation of mental from physical labour which often goes along with an hierarchical structure in the organization of work. That mental work often has a higher social status than physical work cannot be overlooked in a paper dealing with mathematics training for work.

In this chapter, we comment first on the way mathematics is linked to work. We then describe ways and institutions which give training in mathematics for work, and go on to comment on the changes due to the introduction of new technology into the world of work. Some conclusions follow and a list of important references rounds off the chapter.

What mathematics for work?

If the nature of work is known, it would seem easy to describe the necessary mathematics training for the work. One simply looks for the mathematics used in the work and then condenses it into a mathematics curriculum! For developed, industrialized countries this approach was actually tried. It led to interesting results. It was discovered that a direct transcription of the professional use of mathematics into a vocational curriculum of mathematics did not work. One reason for the failure is more or less pragmatic: 'considerable differences . . . were found to exist even within occupations which might be assumed . . . to be similar. It is therefore not possible to produce definitive lists of mathematical topics of which a knowledge will be needed in order to carry out jobs with a particular title' (Cockcroft Report, 1982, p. 19).

A more deeply rooted problem seems to be the identification of the uses of mathematics on the job: mathematics at work is usually *not* the purpose of the work (with the minor exception of some thousands of mathematicians engaged in mathematics research). So it is no surprise to read that 'it was sometimes difficult to establish exactly what mathematics was involved in the task. The problem was to separate what was being used from what we imagined was being used or from what could have been used'. (Mathematics in Employment, 1981, p.8). To use mathematics at work is just one means of coping with the professional situation. The person using it tends to forget about the mathematics if she or he becomes accustomed to using (even complicated) mathematical knowledge.

Besides this subjective side of the story, there is an objective difference between 'mathematics as a discipline' and 'mathematics used at work'. Mathematics as a discipline clearly limits its purpose to that of producing knowledge (e.g. by stating the axioms and using logical principles to derive the consequences). By this procedure, it implicitly defines what is included in 'mathematics'. It is quite otherwise in 'work'. *Everything*, mathematics or not, which helps to cope with a given professional situation, is taken into consideration at least in principle. If a solution to a professional problem is found, it will be used and improved upon as may be possible. Any connections to related theoretical knowledge are often cut off and forgotten as, for instance, the whole body of mathematical knowledge lying behind a certain mathematical proposition which may have been helpful in the given situation. Professional solutions tend to be transformed into mechanical procedures which can be initiated and need not be understood. An illustrative example and a more systematic elaboration of this transformation has been described by Damerow (1984, pp. 11-16).

At a first glance, these considerations seem relevant only to the formalized part of work in industrialized, developed countries. But they are important also for the ever larger number of those who live in rural and/or developing countries. As Broomes put forward quite convincingly, goal one of mathematics for rural development can be formulated as 'the preparation of citizens as users of mathematics', specifying the 'mathematical knowledge needed [as] . . . arithmetic, algebra, geometry, statistics and logic' (Broomes, 1981, pp. 51-2). He himself points to the importance of the contexts in which mathematics is used, so confirming the above analysis of mathematics at work. The examples given by Gerdes (1985, pp. 16-18) in a paper on 'Conditions and Strategies for Emancipatory Mathematics Education in Underdeveloped Countries' show in an impressive way that the topics named by Broomes should not be seen in the same light as is the case in industrialized, developed countries.

Mathematics for work would thus apear to be a more complex phenomenon than the axioms, the statements and the proofs of the discipline itself. Indeed, the structure of the discipline is inappropriate to decide and to describe the contents of a curriculum which aims to provide mathematics training for work. Mathematics for work is closely linked to and even mixed in with problems and situations outside mathematics. These situations have to include certain structures which can be imitated or simulated by mathematics. They can, as we say, be 'mathematized'. The decisive factor in this mixture is the non-mathematical problem, the problem which often obscures disciplinary, mathematical contexts because they are irrelevant to the management of the work-situation.

Mathematics training for work

From the picture of how mathematics is used at work, we now proceed to the explicit theme of the chapter: mathematics training for work. Broadly speaking, there are two types of mathematics training for work: one taking place in special training institutions (such as 'technical' or 'further education colleges') and one taking place at work 'on the job'. These two types of training are closely linked with the social status of the job for which the training is a preparation. The higher the social status of the job and the higher the job is located within the hierarchy of employment, the more probable it is that the training will take place in a full-time institution away from the place of work. For mathematics, the separation from work normally leads to a more disciplinary, more theoretical approach than with mathematics learnt on the job. I will elaborate these statements below by pointing to examples and giving details.

Mathematics training at work

Mathematics training at work does not normally centre around mathematics as such, but is used to cope with a given, professional task. The use of the mathematics is only one part of the professional prcedure (and often a minor one) which tends to become a routine. The procedures usually rely on mathematical techniques taught beforehand at school, such as basic arithmetic, percentages and the rule of three, the list given by Knox (1977) as the 'core' of the mathematics used at work, and which can be compared to the analysis made by Dawes and Jesson (1979). The training will not separate the mathematical part of the procedure from the overall task, but will present the mathematics integrated with, or sometimes even hidden in, the technical or administrative procedure. Consequently, the mathematical knowledge and the background to the solution will be obscured by the special features of the working procedure. This often hinders the learner from even realizing that she or he is using the mathematics taught at school. Here is an example:

> The rule of three, percentages the calculation of interest and the use of proportions are well-established elements of business calculations. The rule of three especially is said to be the most commonly used technique in commerce mathematics. This technique, for instance, is widely used to calculate costs in private enterprises. The fundamental problem of calculating in industry is that of calculating the prime cost of a particular product. Besides the costs which are directly attributable to making a given product (the costs of material, etc.), the overhead costs of the organization have to be distributed over all its products. This is done by awarding the overhead costs to the products in proportion to some fixed basis directly attributable to the products. If the algorithm to be used in the calculation and the necessary information on overhead costs are fixed, the calculation of prime costs can be reduced to mere use of the rule of three and percentages. If, in contrast, a real insight into what is calculated and reasons for the calculations are intended, proportionality must be understood. The linear dependance of the overhead costs distributed to a given product is presupposed as a theoretical tool (for instance, the linear relation to the cost of the material used).

With the example given above, we come to a further aspect of mathematical use and training at work. This is the paramount importance of the *means* used at work and the need to learn what these are during the training. For instance, calculating the prime cost of a product will not be done by paper and pencil without any prescribed method. It will be done according to fixed procedures. These may use work forms especially designed for the purpose or they may use machines which incorporate the algorithm of the calculation like the spreadsheets of modern computer software. The theoretical knowledge which justifies the procedure (which often partially relies on mathematical knowledge) is not needed to carry out the procedure. Nor may it be taught as part of the mathematical training given at work.

In some countries, training at work is supported by a certain type of training which, in principle, belongs to the institutionalized training, but which needs to be mentioned here. These are the sandwich courses which alternate training at a college with work and training on the job. This type of training can be found, for instance, in the United Kingdom and in the

Federal Republic of Germany (Straesser, 1985). Most of the week is spent at work, where training may be provided, but one (or sometimes two) days of the week will be spent on training in the technical/vocational college. Sometimes this 'dual system' of vocational training will condense the college attendance into periods of six weeks to three months, leaving the rest of the year for continuous on-the-job training. The mathematical part of this 'sandwich' type training normally takes place in the college and leads to the consequences described in the next part of the chapter.

Mathematical training in technical and vocational colleges

Training in an institution away from the place of productive work necessarily implies certain consequences. What is to be learnt is divorced from its everyday context of use. It is transformed into an explicit or a hidden curriculum and loses its links to the technology and organization of work. But there are potential gains too. Implicit foundations and justifications can be explained, discussed and evaluated. Structures which are common to different work procedures can be explained and theory can be elaborated. As a result, the trainee may develop from being a mere operator of vocational procedure to a well-informed, self-reliant and creative expert in professional situations. The mathematics which underlies a special situation may be perceived, mastered and even adapted to changing circumstances, if the training is appropriate. The college setting can have negative effects on mathematics too. This has been shown by empirical studies. Street vendors in Northeast Brazil did quite well in their usual place of work when they had to calculate prices and give change. But when they were placed in a formal, college-type setting, their performance on comparable tasks was substantially inferior, and the 'procedures were qualitatively different' (Carraher et al., 1986, p. 88; Schliemann, 1986, p. 93). Mathematical training in technical and vocational colleges can have all ingredients mentioned above. Here is an example.

The use of formulae is very widespread in technical work like metalworking, electricity, building and other constructive sectors of economy. A closer look at them shows that formulae are a very effective and short way of presenting models of qualitative and quantitative relations between physical and technical magnitudes. They can be powerful mathematizations of vocational situations, and can explain the ways and size of change when some of the variables are altered deliberately (e.g. changing electromotive power in Ohm's law). They can even be used to gain insight into a technical field of application. For example, the law of the lever can be used to calculate the forces a railway bridge has to stand. Training in the use of formuale can be given at different levels, from merely inserting numerical values into given formulae, to easy manipulation, derivation and evaluation of formulae using the dimensions of the magnitudes as an additional check on the solution (Harten, 1985; Straesser, 1981). Manipulation of formulae may even be outdated in the near future with a more widespread use of software packages, including symbolic algebra.

If one looks into the contents of the mathematics offered in college for the future qualified worker, one finds that the 'core' (basic arithmetic, percentages, rule of three, etc.) seems to be taught in most developed, industrialized countries. In addition to the core, specific topics are added for future metal workers, electrical and electronics apprentices and some jobs in the building sector. They include basic algebra (especially the manipulation of formulae), reading and interpreting tables, diagrams and graphs, and some more advanced techniques. Commerce apprentices are given training in business calculations and some descriptive statistics. Geometry and technical drawing play an important role in the construction trades for future workers in wood, metal and for bricklayers. (Straesser, 1980). From developing countries less information is forthcoming on what is taught in 'vocational' colleges if, indeed, there are any. But it would appear that in some colleges mathematical topics are taught without reference to their use, whereas in other places these are only taught as an integrated part of the vocational training.

The paramount difficulty in the college training of future qualified workers is conveying to them the close relationship between the practical situation and the mathematics called for. They find it hard to select the appropriate formula, and/or the technique which a given practical situation requires; this problem appears to be widespread (Ploghaus, 1967, p. 524). Most of the mathematical skills needed in technical and vocational education situations are taught in compulsory education. The problem is getting the learner to apply them (Cockcroft Report, 1982, pp. 18 et seq.). Technical and vocational colleges seem to overlook the need for learning to *use* mathematics in practical situations. They put too much stress on teaching topics their students, for the most part, already know.

If we now 'climb the ladder of qualification' and consider trainees with a qualification for university entrance, the situation in mathematics training for work becomes somewhat paradoxical. Mathematical training for this level of qualification is widespread and hardly anybody would deny its necessity. But empirical studies show quite convincingly that future employers of this level of qualification are more interested in recruiting employees with a qualification in mathematics than in any specialized training for specific jobs (Borovcnik et al., 1981, pp. 50-8, 78-86). For entrants with this level of qualification, the employers seem to be perfectly happy with a disciplinary approach to mathematics teaching at college (basic and linear algebra and calculus). They ask only for application-oriented training for those going into the industrial parts of the economy, together with some preparation in statistics and computer use. They seem to be not too interested in direct training for work itself. Such knowledge and skills as are needed at the workplace are imparted on the job.

The paradox could be interpreted thus: as mathematics at work is so closely related to what goes on on the shop floor, there is no point in giving vocational training in mathematics for work performed elsewhere. The way a problem is seen, formulated, handled and solved remains so near to its source that mathematical concepts and procedures are only used *ad hoc*. This part of the interpretation can explain why colleges give no direct mathematics training for work. But why is mathematics taught at all? The answer to this question must be sought in hope, hope that certain general skills and attitudes are required as a by-product of mathematics. The ability to transform a problem into mathematical terms, to devise a simple procedure to cope with a situation and to interpret the results deriving from a given procedure. In short, the ability to model a given vocational/professional situation with the help of mathematics presupposes learning, knowledge and frequent use of mathematics. In addition to these abilities, related as they are to mathematics, there are widespread beliefs that mathematics also fosters certain general skills and abilities such as accuracy, perseverance, self-criticism and logical thinking. As there is little evidence for this belief, such expectations can be termed only as hopes! Now a digression:

'The responsibilities and the problems associated with training for work in colleges can be very well exemplified by a closer look at the French institutions of technical and vocational education. Since the French revolution, training for work has traditionally taken place in full-time colleges, where different lines of training are offered. These different types of training are ordered in a clear hierarchy from a very broad, low qualification (almost a special way to wait for an unskilled job!) up to highly specialized professional qualifications which also allow entry to university. A closer look at the mathematics curricula gives the impression that the teaching of mathematics is rather formal. The disciplinary approach is the main characteristic of this training. Vocational and professional procedures and knowledge play only a subordinate part in the mathematics training at French technical and vocational colleges (*Lycées d'enseignement professionnel*). One could even believe that selection, not qualification, is the main aim of mathematics training in these colleges' (Braun and Straesser, 1980, p. 154).

To end this section on mathematics training for work a certain contemporary feature must be highlighted. This is that in some countries 'mathematics for work' is taught, but not obviously to equip the learner. With unemployment rising in industrialized countries, mathematics is often taught with a view to achieving success in entrance tests set by employers. Usually, these tests are irrelevant to the needs of any future work. The rising number of unemployed youngsters itself leads to more students attending college. They attend both to avoid unemployment and in hope of improving their qualifications for work. In such circumstances, mathematics training for work plays an important role in the selection, not the qualification, of future workers.

Technology and mathematics for work

One major topic of the current debate on mathematics education and mathematics training for work is 'the influence of computers and informatics and its teaching', the title of the first international study of the International Commission on Mathematical Instruction (ICMI, 1985). If we take computers and new communication technologies as examples of the fast-developing technology and organization of work, we are at the heart of one 'key issue' of mathematics training for work (Straesser and Thiering, 1986).

Earlier, we considered the importance of the means and the tools used at the workplace. The widespread use of handheld calculators and computers obviously influences the use of mathematics in work. The growing division of labour tends to lower the level of mathematical competence demanded for employees because it concentrates the planning and controlling of the work and the associated mathematical competences into the hands and minds of a few experts. In this connection, Fitzgerald (1985) gives a detailed analysis of the situation in the United Kingdom. Here are some examples.

In business and administration in particular, the computer is taking over much of the work of producing tables and doing the accounting and the internal calculation of companies. But studies on the introduction of computers into this economic sector show that more structural background knowledge is needed with the advent of computers (Baethge, 1982). This can be seen as a growing need for more abstract, if not mathematical, concepts in order to understand and to take part in the running of a company. To be more specific: the growing use of business software (spreadsheets, etc.) may create a demand for more abstract and mathematical instruction or education.

A different example from industry points in the same direction. The use of computer-assisted-design (CAD) techniques and computerized-numerically-controlled (CNC) machines seems to require at least some understanding of co-ordinate geometry and linear algebra. In the Federal Republic of Germany, this was formulated as a special research and development problem by the central institution responsible for technical and vocational education (Buschhaus, 1984). In this case, mathematical knowledge seems to be needed as a structural background for the specific use of computers. Furthermore, the need for singular calculations of special values seems to diminish. To sum up, the vocational use of computers seems to lower the importance of arithmetic and make more important structural, mathematical knowledge.

The above can be formulated in more general terms. With the growing sophistication of technology and of the organization of work, learning at work is becoming more and more difficult. This is partly due to financial reasons, since the break-down of a machine produced by an inexperienced worker or learner leads to more expense with every advance in technological development. Likewise, technological development increases the cost of the machines operated by a worker and makes for an increase in the financial loss when errors are made. This fact makes for increasingly serious problems with training-on-the-job.

This is also true for organizational reasons, because the growing complexity of interrelated and automatized work obscures the purpose of an individual's particular job. This impedes identification with the job and with effective and pertinent learning. Alienation from daily work, due to the increasing opaqueness of its reasons and purposes, tends to impede learning at work. A better understanding of mathematics and of mathematical structures may make for a better understanding of the work in progress and of the benefits of technological and organizational change. If so, mathematics could be even more important for technical/vocational education in the future than it is now.

Another aspect of technological and organizational change at work is opened up with the introduction of CNC machines in industry. Certainly in the Federal Republic of Germany this introduction shows two quite different effects:

Writing the programs needed to operate these machines (e.g. a turning lathe) is done in a laboratory or an office by engineers. Flexibility and mathematical knowledge are not needed by the machine operators.

Sketches and/or technical drawings of the product to be made are given to qualified workers who do their own programming or optimization of programs and on the shop floor at the CNC machine. They have to execute and supervise production and require a knowledge of co-ordinate geometry in three dimensions in order to do their programming.

Both patterns occur in industry in the Federal Republic of Germany and they obviously lead to contrary demands for mathematics education.

Conclusion

In different countries, the system of providing technical and vocational education varies from full-time attendance in technical colleges (e.g. France) to isolated or part-time activities in private enterprises or colleges (e.g. the United Kingdom and the Federal Republic of Germany for some levels of qualification). Some countries do not even have any institutions which could be called technical or vocational colleges.

Mathematics training for work necessarily has to cope with the fact that the purpose of work is not usually to do mathematics. If needed at all, it is only one (sometimes minor) means of coping with a situation which is often deeply interrelated with other factors and so cut off from its disciplinary, scientific context. Training in mathematics for work must accept this fact. It can be taught in the context of the work in hand, which obscures its mathematical context, or it can be taught apart from the work place in a college, classroom or some other pedagogical institution. In a college setting, learning to use mathematics in a practical situation seems to be the major difficulty in giving credible mathematics training for work. Consequently, the use of professional aids to cope with a situation (tools, forms, machines, tables, graphs, pocket calculators, computers and professional software) assumes a role of utmost importance in providing effective mathematics training for work. The converse is also true. The quality of the education in mathematics deeply influences the technology and the organization of work which a country can use.

Two general problems tend to bedevil the efforts of those concerned with giving training for work: the widespread disapproval of technical and vocational training which is seen to be inferior to 'general' education and the fact that, in most societies, the social and economic process does not adapt itself readily or willingly to the production and distribution of goods and services and so to the preparation for this process in technical and vocational education. This problem manifests itself in the shortage of competent teachers and in the lack of adequate curricula and teaching/learning materials (Unesco, 1978, pp. 99 et seq.). Shortage of competent teachers is also a major problem with providing mathematics training for work.

References

BAETHGE, M. 1982. Auswirkungen des Einsatzes der Informationstechnologien auf die Qualifikation und Ausbildung kaufmännischer Angestellter [Effects of the Use of Information-technologies on Qualification and Training of Business Employees]. *Log in* (Kiel, Institute for Science Education), Vol. 2, pp. 15-18.

BOROVCNIK, M.; DÖRFLER, W.; KAUTSCHITSCH, H.; MALLE, G.; PESCHEK, W. 1981. *Mathematik in der beruflichen Praxis von Abiturienten* [Mathematics in the Professional Practice of College Graduates]. Vienna/Stuttgart, Hölder-Pichler-Tempsky/Teubner. (Schriftenreihe Didaktik der Mathematik, Vol. 5.)

BRAUN, H.-G.; STRAESSER, R. 1980. Mathematik in der beruflichen Bildung in Frankreich [Mathematics in Technical and Vocational Education in France]. In: *Zentralblatt für Didaktik der Mathematik* (Stuttgart), Vol. 12, No. 4, pp. 150-6.

BROOMES, D. 1981. Goals of Mathematics for Rural Development. In: R. Morris (ed.), *Studies in Mathematics Education*. Vol. 2, pp. 41-59. Paris, Unesco.

BUSCHHAUS, D. 1984. Auswirkungen der CAD-Technologie auf die Qualifikationsanforderungen in den technisch-zeichnerischen Ausbildungsberufen [Effects of CAD-technology on Qualificatory Demands for Trainees in Technical Drawing]. In: N. Meyer and H.R. Friedrich (eds.), *Neue Technologien in der beruflichen Bildung* [New Technologies in Technical and Vocational Education], pp. 134-9. Cologne, Verlagsgesellschaft Schulfernsehen mbH & Co. KG.

CARRAHER, D. W.; CARRAHER, T. N.; SCHLIEMANN, A. D. 1986. Having a Feel for Calculations. In: International Congress on Mathematical Education (5th), Adelaide, Australia, 1984, *Mathematics for All*, pp. 87-9. Paris, Unesco. (Science and Technology Education Document Series, 20.) (ED-86/WS/84.)

COCKCROFT REPORT. 1982. Committee of Inquiry into the Teaching of Mathematics in Schools. *Mathematics Counts: Report of the Committee of Inquiry . . . under the Chairmanship of W.H. Cockcroft*. London, HMSO

DAMEROW, P. 1984. Mathematikunterricht und Gesellschaft [Mathematics Education and Society]. In: H.-W. Heymann (ed.), *Mathematikunterricht zwischen Tradition und neuen Impulsen*, pp. 9-48. Cologne, Aulis-Verlag Deubner. (Institut für Didaktik der Mathematik (Bielefeld): IDM series: Untersuchungen zum Mathematikunterricht, Vol. 7.)

DAWES, W. G.; JESSON, D. St. 1979. Is there a Basis for Specifying the Mathematical Requirements of the 16 Year Old Entering Employment? *International Journal of Mathematical Education in Science and Technology* (London/Philadelphia, Taylor & Francis), Vol. 10, pp. 391-400.

FITZGERALD, A. 1985. *New Technology and Mathematics in Employment: A Summary Report*. London, Department of Education and Science.

GERDES, P. 1985. Conditions and Strategies for Emancipatory Mathematics Education in Underdeveloped Countries. *For the Learning of Mathematics; An International Journal of Mathematics Education* (Montreal), Vol. 5, No. 1, pp. 15-20.

HARTEN, G. V. 1985. Formeln in der Berufsschule und in der Sekundarstufe I des allgemeinbildenden Schulwesens [Formulae in Technical Education Colleges and in Lower Secondary General Education], *Mathematica Didactica* (Bad Salzdetfurth), Vol. 8, pp. 179-94.

HORNBY, A. S.; GATENBY, E. V.; WAKEFIELD, H. 1960. *The Advanced Learner's Dictionary of Curent English.* 11th ed. London, Oxford University Press.

ICMI (INTERNATIONAL COMMISSION ON MATHEMATICAL INSTRUCTION) (ed.). 1985. *The Influence of Computers and Informatics on Mathematics and its Teaching Supporting Papers.* Strasbourg, Institut de Recherches sur l'Enseignement des Mathématiques (IREM). (The book can be ordered from the Université Louis Pasteur, Université de Strasbourg I, 4 rue Blaise Pascal, 67070 Strasbourg Cedex, France.)

KNOX, C. 1977. *Numeracy and School Leavers A Survey of Employers' Needs.* Sheffield, Sheffield Regional Centre for Science and Technology. (Paper, No. 12.)

Mathematics in Employment (16-18). 1981. Bath, University of Bath. (Report.)

PLOGHAUS, G. 1967. Die Fehlerformen im metallgewerblichen Fachrechnen und unterrichtliche Maßnahmen zur Bekämpfung der Fehler [Types of Mistakes in Mathematics Instruction of Metalworkers and Ways to Fight the Mistakes]. *Die berufsbildende Schule* (Wolfenbuettel), Vol. 19, pp. 519-31.

SCHLIEMANN, A. 1986. Mathematics Among Carpentry Apprentices: Implications for School Teaching. In: International Congress on Mathematical Education (5th), Adelaide, Australia, 1984, *Mathematics for All*, pp. 92-5. Paris, Unesco. (Science and Technology Education Document Series, 20.) (ED-86/WS/84.)

STRAESSER, R. 1980. Mathematik in der Teilzeitberufsschule [Mathematics in Part-time Vocational Colleges]. *Zentralblatt für Didaktik der Mathematik* (Stuttgart), Vol. 12, No. 3, pp. 76-84.

——. 1981. Gleichungslehre in der Berufsschule [Teaching Basic Algebra in Vocational Colleges], *Mathematica Didactica* (Bad Salzdetfurth), Vol. 4, pp. 105-13.

——. 1985. *Mathematics in Technical and Vocational Education in the Federal Republic of Germany (FRG).* Bielefeld, Institut für Didaktik der Mathematik [Institute of the Didactics of Mathematics]. (Occasional Paper, No. 62.)

STRAESSER, R.; THIERING, J. 1986. Action Group 7: Mathematics in Adult Technical and Vocational Education. In: M. Carss (ed.), *Proceedings of the Fifth International Congress on Mathematical Education* [Adelaide, Australia, 1984]. Boston/Basel/Stuttgart, Birkhäuser.

UNESCO. 1978. *Developments in Technical and Vocational Education: A Comparative Study.* Paris, Unesco. (Trends and Issues in Technical and Vocational Education, 1.)

10. Family Math

Jean Kerr Stenmark

> 'We haven't laughed like this together in a long time!'
>
> 'You can't imagine what ten minutes a day can mean to your child - homework without frustration and tears! It enhances your child's and your own understanding. She gets math in a nonthreatening, nonpressured way - the way that we read to her'.

These are comments made by people who have taught or attended Family Math classes, where parents and children come together, learning new ways to do, to understand and most of all to enjoy mathematics. As Virginia Thompson, director of this project, says: 'When we want to help children with reading, there are lots of things we can do: read to them, look at picture books, go to the library - all fun and enjoyable. What about helping with math? All most of us can think of is to help our children learn the multiplication tables by using flash cards or insist that they do their homework before they go out to play'.

Family Math is a programme which has been developed by the Lawrence Hall of Science at the University of California, a public science museum with a mission to increase public understanding and enjoyment of mathematics and science. It provides training and teaching materials for classes which may be taught by a teacher, parent or community worker in a school, church or home. They give parents, together with their children, opportunities to develop their problem-solving skills and to understand the mathematical concepts their children need to know. Families involved enjoy doing mathematics to further education and work, through meeting women and men from the community who are working in math-based fields.

It is acknowledged that some groups of students are more likely than others to stop taking mathematics as soon as they can; such students need encouragement to persist in mathematics, to see the relevance of math to their future lives, and tc develop confidence in their mathematical ability.

Dropping out of mathematics too soon may severely reduce future options in higher education or in the work force.

Since 1977, a Lawrence Hall project, Equals, has provided activites and methods to help teachers and their students build success in mathematics, with special focus on problem-solving, understanding rather than memorizing, co-operative learning and the use of manipulatives. Workshops model good teaching practices, such as using calculators and computers as tools, frequent change of pace and regrouping, use of the language of mathematics, and keeping mathematics journals. Teachers can give their students a myriad of opportunities for thinking mathematically.

Teachers and other educators who attended this project, aware of the importance of students' experiences outside the classroom, asked for ideas and materials for parents who wanted to help their children with mathematics. From this came Family Math in 1981.

With funding from the United States Department of Education, the first year of development included creating or gathering activites and conducting courses for parents and children in nearby urban school districts. Training workshops followed in the next and subsequent years for those who wanted to teach the course themselves. These same workshops served as models and training sessions for another group who wanted to become 'trainers', giving their own workshops in their own locations to create new groups of class leaders.

Thus, the programme has three components: classes taught for parents and children together, workshops to prepare parents and other adults who want to present classes, and training sessions for those who want to teach new class leaders.

Classes are usually taught by grade levels (for example, K-2, 3-4, 5-6 or 7-8), meeting for about two hours a week, over four to six weeks. The mathematical content covered is broad, including probability and statistics, logical thinking, geometry and spatial reasoning, algebra and arithmetic, measurement, functions, patterns, and so on, in support of what is considered a good school curriculum for children from 4 or 5 years to teenagers.

In the suggested curriculum, leaders, parents and children solve problems together, learning together as they go. Both projects stress the importance of the problem-solving process rather than finding answers. Learning to think about mathematics is the key. As one parent said, 'I know I don't have to pressure my kids to have answers or to be right all the time'. And the same applies to class leaders.

Family Math materials are devised to work with children (and parents!) from pre-school through high school, and with people from diverse ethnic and socio-economic backgrounds. A number of professional mathematicians have been involved in the programme and all have found the mathematics sound. Most of the game-like activities in a class are planned to develop problem-solving skills, to build understanding with 'hands-on' materials and to practise talking about mathematics.

By problem-solving skills, we mean ways in which people learn to think about a problem, using stategies such as looking for patterns, drawing a diagram, working backwards, working with a partner, looking at one part of the problem at a time, or even letting the problem sit in the back of your head for a while! Having a supply of strategies relieves the frustration of not knowing how or where to begin. Learning to take the risk of working on a difficult problem and sticking with it is vital. More strategies result in greater confidence and willingness to tackle new problems, and therefore better problem-solvers.

'Hands-on' materials are concrete objects such as beans, pennies or toothpicks used to help children understand what numbers and space mean and to help all of us solve problems. Traditionally these materials are used in the early elementary years and pencil-and-paper become the rule after second or third grade. This is unfortunate since much of mathematics can be best explained and understood using the tools of manipulative materials and models. In fact, many research and applied mathematicians do just that.

Talking and writing about the problem-solving process is an important element. strategies and procedures put into words will clarify meaning and help us learn new ideas. Parents are particularly encouraged to develop a repertoire of leading questions, providing motivation for their children to think for themselves. Parents in these classes are also given overviews of the usual mathematics topics at their children's grade levels, with explanations of how these topics relate to one another. The school curriculum begins to make sense to parents when they know how one concept builds the base for the next.

Other activities include some that link mathematics to careers, discussing its future use in jobs and inviting role models from a variety of occupations to come and talk to the families. Many role models explain that they didn't always get good grades in math, but they stuck to it and are on the way to reaching their goals. Almost every role model will talk about the importance of problem-solving and the need to learn to work in groups, the everyday realities of the working world. This is a very powerful motivation for students to contine taking mathematics, even when it becomes more difficult.

Parents who take part have a rare opportunity to learn about their children. Said one: 'I could see the dynamics of the family. School psychologists should stop spending so much time testing snd come to Family Math'. They will also learn a lot about the real meaning of mathematics. An instructor observed: 'Learning that math was not arithmetic was the biggest breakthrough parents made, also that drill homework is not beneficial.' For some parents, especially those from a low-income, minority or non-English-speaking community, this can be a great deal more than a user-friendly math programme. It can be a means by which parental involvement in their children's schools will begin. The emphasis on prevention of fear or failure, on enjoying activities and on working together make the programme a powerful vehicle for parent-school communication, a benefit for schools as well as for parents.

Parents who share learning with their children, provide a supportive environment, and model risk-taking, persistence and other problem-solving behaviours will feel personal gain through their strong positive impact on their children's educational success. A secondary but equally exciting benefit is demonstrated by parents themselves, many of whom are ecstatic at enjoying mathematics for the first time in their lives. Some go on to take serious mathematics courses. Others become teachers within the programme, or begin to help the teachers at school do math projects. To date, an estimated 15,000 families have participated.

In addition to organized effort, many individuals will be able to teach classes or simply work with their own family by using the Family Math book (Stenmark, Thompson and Cossey, 1986), a collection of activities and ideas for use by parents. Materials suggested for use in the activities are inexpensive or in many cases are free 'found' items which should be available in less-mechanized cultures. Most of the teaching materials have been translated into Spanish.

Instructors sometimes prepare and teach subsequent courses different from those taught in the first year or two, branching out occasionally into other topic areas. Thus growth takes place not only by doing the same thing many times, but also through broadening and deepening the use of the programme.

Reference

STENMARK, J. K.; THOMPSON, V.; COSSEY, R. 1986. *Family Math.* Berkeley, Calif., Lawrence Hall of Science, University of California.

Part IV

A case-study

11. Out-of-school mathematics in Colombia

Mary Falk de Losada
Ricardo Losada Marquez

While the teacher-student ratio in schools ranges from 1 to 20 to 1 to 60, traditional classroom education cannot be expected to be particularly effective, as the more students there are per teacher, the lower their academic success rate. This is even more true in schools where there is a shortage of qualified staff. In such circumstances, out-of-school mathematics programmes can provide more opportunities for more young people not only in individual schools but on a regional and a national scale. By making use of the better qualified teachers, this kind of activity has an extraordinary multiplier effect that has not as yet been fully exploited. Such provision should not be regarded as another cold technical problem of how to get the best out of a system, but as an obligation that a society has towards its young people, the obligation to open the doors of the future to them by providing an education geared to their interests, skills, aspirations and talent.

Establishing tradition

The first university mathematics course was launched in Colombia in 1956 and post-graduate mathematics courses started in 1968. Until then, traditional mathematics education in Colombia centred on primary and secondary level education and, at the tertiary level, on the training of engineers and architects. Interest in mathematics as a science in Colombia is very recent when compared with many other countries in the world.

This situation was conducive to the introduction of new approaches to the study of mathematics and, at the same time, made it a matter of vital necessity that they should succeed. Out-of-school mathematics activities grew out of the country's participation in mathematics Olympiads, but they involve more than a purely competitive approach to mathematics.

New approaches

Each country's special situation calls for special approaches. In Colombia, there are some very isolated areas; there is also a dearth of well-qualified teachers and a rapidly growing student population. The education system, based on official syllabuses, attempts to provide all students with the same courses and opportunities, regardless of their interests or particular skills. These official syllabuses specify the ground to be covered in every year. For some pupils they are too demanding; but for others they are extremely limited in their objectices and aims when set against these students' capacity and potential.

A great deal has been done in Colombia to reach underprivileged and isolated groups in the population and to offer them the opportunity to continue and complete their primary, secondary and university education. Colombia has for years provided primary and secondary education through radio and television, and it has recently put tremendous efforts into distance education at the university level.

In this paper, we shall concentrate on programmes designed for those students who show a special interest in the study of mathematics and/or an outstanding aptitude for the subject. Numerous out-of-school experiments have been organized with a view to meeting their special requirements and to building up the future of science in Colombia.

The programme of mathematics competitions was established in 1981 as an incentive for young people. At the same time, a series of important back-up activitities was organized to provide pupils with an opportunity for self-improvement and personal development. Among the main activities may be mentioned the following:

A Sunday newspaper column on mathematics.

The publication of pamphlets containing problems which complement the text books.

The preparation of a bi-monthly review to provide source material for the activities of mathematics clubs.

Giving guidance to secondary schools and other bodies wishing to start mathematics clubs.

The organization of regional workshops.

The newspaper

A most interesting feature of Colombian out-of-school mathematics is the weekly column of problems and solutions, providing both education and information, which is written by the organizers of the Colombian Mathematics Olympiads and is published in the Sunday edition of *El Espectador* (Bogotá). On almost every Sunday since November 1981 this fascinating column has reached some 300,000 Colombian homes, giving nearly 2 million people the opportunity inexpensively to improve their mathematics a little every week while getting the sports and other news.

The original purpose of the mathematics column was to inform the public about national and international competitions and there is no doubt that this was a very useful way of publicizing them. But, in addition to making pupils and teachers alike aware of the Olympiads, the content evolved to the point where it was performing a series of broader functions, some of which we shall illustrate in detail with quotations from reader's letters.

As well as providing information on the dates, the entrance formalities and other details of the various competitions and in competitions, the column began to publish problems set in earlier national competitions and in competitions of a similar standard conducted in other countries. This development served the specific purpose of acquainting pupils and teachers with the type of problems entrants would encounter in the competition, the mathematical content covered by them and their level of difficulty. In other words, the newspaper column was used to explain the venture to the educational community in preparation for the first national Olympiad. This was undeniably helpful in dispelling the uncertainty that might have caused students to hesitate or to decide not to take part. Questions set in various competitions are still published frequently to encourage even greater interest.

It was not long before the column broadened its scope again by including general articles on mathematics, so forming a collection of essays which focused attention first on one particular type of problem or on one particular method of finding a solution and then on another.

Mention should be made of the technical problems involved in writing on mathematics in a newspaper that has no access to mathematical symbols and no typesetter specializing in mathematical texts. Diagrams and the more complicated mathematical expressions are photographically reproduced within the column. Since, however, photographic reproduction is very costly, we have to devise our problems and their solutions so as to cut down the use of complicated symbols to a minimum. Consequently, it is only fair to say, mathematical symbolism has suffered a serious reverse at out hands; but few complaints have been received. Sometimes a misprint creeps into the column, but on the few occasions that this has occurred readers have detected it, corrected it and solved the problem in its intended form.

The column offers a wide variety of 'teasers' (by which we mean problems that stimulate particular interest or curiosity). Some of them are original, but most have been taken from competitions held in different countries, from the international Olympiads or from published collections of problems. In the beginning we were not particularly circumspect in our choice of problems: mathematical interest was the criterion. But this policy had to change because some of the solutions were so long, or included so many symbols, that they could never be published. However, even though such problems have to be ruled out, there is still an inexhaustible fund of suitable problems.

As space is very short, precision and absolutely clear and concise drafting is essential. This calls for a very special style. At the same time, a wide variety of themes and approaches is needed in order to maintain the readers' interest. The column has dealt with such varied subjects as well-known mathematical errors, maths and humour, the method for considering factors in the solution of Diophantine equations, methods used in recursion, maxima and minima without calculus and calendar problems for the new year.

It is absolutely vital to present topical problems if they are to attract the reader's attention and create an enthusiastic and optimistic mood. Perhaps the best way of explaining this is to give examples in the form of facsimiles of the sections on the use of arguments of parity to solve problems and of the section devoted to alphanumerics.

inset 1


Olimpiadas Colombianas de Matemáticas

Métodos poderosos y sencillos

Olimpiadas de Matemáticas

Para el nuevo año, ¿un simple almanaque?

Olimpiadas Colombianas de Matemáticas

Alfamético

Scientific criteria are also important in determining the themes to be dealt with. We have written on a wide range of topics that are not particularly well known in Colombia, and, most importantly, on some that do not feature in the textbooks or in the reference-books available to Colombian teachers and students. The letters we refer to below have been chosen from a large pile of correspondence which makes it clear that the readers, as well as the authors, of the Sunday column realize that it serves as a supplementary book of reference.

One young reader wrote to us as follows: 'There was a time when books were a rare treasure, the property of a tiny élite. You have carried out a democratic revolution in education by giving everyone access to the means of knowledge'.

As gratifying, though less dramatic, was the letter from an education inspector in the Department of Antioquia who said that he used the column for his lectures on mathematics teaching methods during school visits to teachers in his area. We also received a solution to a Diophantine equation problem from a teacher in Sincelejo in the Department of Sucre which was accompanied by a letter which contained the following:

> Various end-of-year chores diverted my attention somewhat from the fascinating problems in *El Espectador*, but today I got down to it and am sending you my solution . . . I confess that the problem fascinated me so much that it made me work out from scratch the methods used to solve linear Diophantine equations with unknown quantities, which had been completely buried in the back of my mind.

It is very satisfying to be told that the column is making a contribution to teacher training by prompting teachers to improve their own command of the subject. At the same time, there is a fear that the rote learning that constitutes mathematics teaching in many Colombian schools, the trivialization of the subject by the mechanical repetition of definitions and the premature and unrewarding formalization of the subject may be emptying the university mathematics and science departments. This fear has inspired an attempt to use the newspaper column to change the direction of mathematics education for Colombian youth and make it a creative activity geared to problem-solving. This objective has to some extent been attained, since pupils and teachers alike have expressed their satisfaction with the focus on problem-solving. They see it as an innovation which provides them with a more palatable approach to learning and teaching mathematics, as the following comments from readers bear witness:

> A reader concerned about mathematics salutes you and congratulates you on having awakened student interest in this subject. In the past, mathematics was regarded as something that only the most intelligent students could attempt; today we realize that this is not so.
>
> Reading Sunday's *El Expectador* I came across the section on the Mathematics Olympiads, which I found extremely interesting, since it enables students like me, studying for the *baccalauréat*, to learn this important subject in an entertaining and exciting way.
>
> Cordial greetings and a word of encouragement for your noble and benevolent work of helping thinking young people. Little by little you are arousing these young people's interest in a science which many people shy away from, thinking it too difficult; however, thanks to your efforts and concern, these hurdles are being overcome and it is now being seen as a source of intellectural satisfaction. I share the view that solving one problem that poses a challenge to our ingenuity is worth far more than completing 300 dull and uninstructive exercises.

Every week we publish some new 'teasers', together with readers' solutions to previous problems. For those who are not directly involved in education, who may be professional people of all kinds, as well as youngsters who have had to leave school early in order to go to work, these problems offer an opportunity either to acquire a taste for mathematics or to keep up their interest. We also receive letters from this kind of person, describing what the column means to them.

> Nearly every Sunday the first thing I look for as I leaf through *El Espectador* is your delightful mathematics section; sometimes I amuse myself by solving the problems. Today I found something on Magic Squares on which there is much to be said. I would be delighted to continue to collaborate on this theme and possibly on other problems.
> Allow me to congratulate you and beg you to continue including these fascinating problems in the section on the Mathematics Olympiads. They are useful Sunday entertainment for those of us who work, and food for thought during the week for students. Today I felt something I haven't felt for some time: that I was a student again. Thank you for an enjoyable afternoon. I enclose my solutions. If they are no good, I shall be able to try again; if they are right, I won't have wasted my Sunday afternoon.

The column serves as a means of continuous self-evaluation for many people; numerous regular readers frequently send in solutions.

All the letters with incorrect solutions are answered with suggestions as to how they could be corrected, and, in this respect, mass communication can be said to offer a valid learning experience. The readers appreciate the system:

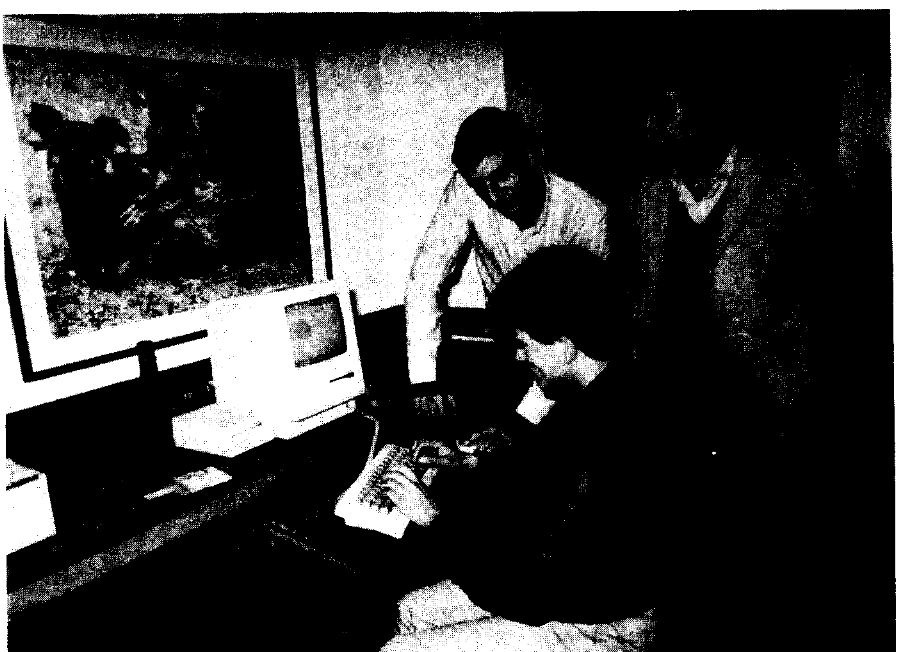
> Thank you for your letter. I am grateful that a scheme like yours caters for people who are not taking part in the Mathematics Olympiads but who follow the progress of the Olympiads through *El Espectador*. I acknowledge my mistakes; I think I rushed into the problem too hastily and did not give it enough thought.
> My sincere thanks for the solution to the problem you sent me. All in all, considerable effort is needed to send you any solutions to the problems that appear in *El Espectador* because when I have the information on bibliographies it is insufficient.

This last comment evokes the problem of reference material. We shall return to it when we describe the publication of booklets of problems and solutions.

Communication with the public through the newspaper has made us appreciate the tremendous public interest in the Mathematics Olympiads. If for some reason the maths column does not appear, readers immediately protest and write wanting to know why. Others expect the competitions to have an impact be difficult to achieve in reality, but the fervour of their conviction is worth mentioning. Here is one of the comments received:

> A few days ago in *El Espectador* I read your wonderful column entitled 'Mathematics Olympiads throughout the country'. I was most impressed by the wide variety of stimulating problems you present which will encourage our entire population to think and argue logically: you are encouraging human perfection.

In Colombia, as in many other countries, some people live in very isolated areas. Correspondence about the column is received from 20 per cent of the country's municipalities. Among the letters received from people who, despite their isolation, maintain a special interest in mathematics, one of the most memorable was sent by a boy from the south of the country. He wrote: 'I should like to be your friend because we have something in common: mathematics.' One of the finalists in the 1982 and 1983 Olympiads prepared himself for the contest by solving the problems in the newspaper and following the explanations. He had to place a special order for the newspaper in the capital of his department since it was not normally delivered to his village. The Sunday edition reached him on Tuesdays. It is possible that the success of this venture was owed, in part, to the fact that the pupil came to study the column without being prompted to do so. Although many teachers have welcomed the maths competitions, and maths clubs, etc., as usefully encouraging and promoting study, there is reason to think that other teachers see the Olympiads and the ingenious problems involved as just another chore, extra responsibility to be avoided if possible.



The magazine *MATICO*

Effective and interesting though mass communication through the press may be, it has its limitations: space is limited; the problems must not be too difficult; and it is vital that the column should appeal to as wide a readership as possible. It promotes the 'flavour' of mathematics in a new way. But the effective build-up of interest and mastery of the subject must be the responsibility of a more specialized publication. The magazine *MATICO; La Revista de matemática joven* (Young Mathematician's Magazine) was founded in September 1982[1] precisely to realize these aims. *MATICO* is written by young people for young people. In the first three years since it was founded, it was edited by people under 20 years of age. Its specific purpose is to provide a constant supply of material to support mathematic clubs.

MATICO is a bimonthly journal of some eighty pages. The problems it sets are divided into sections:

Para romperse la cabeza (Brain-teaser) contains puzzle-type problems and problems whose solution calls more for intelligence than an extensive knowledge of mathematics. It is designed essentially for younger readers who are in the sixth to eighth year of schooling, i.e. the first three grades of secondary education in Colombia.

¿Puede resolverlo? (Can You Solve It?) contains problems for pupils in their ninth to eleventh year of schooling. These problems require more knowledge than those in the previous section, but they are not very difficult and most of the pupils in the target age-group are expected to be able to work them out.

Olimpiadas alrededor del Mundo (Olympiads Around the World) contains a variety of problems that have appeared in the final rounds of mathematics competitions held in various countries throughout the world. This enables the reader to maintain contact with international competitions and measure his or her abilities against those of the best pupils in other countries.

Problema del mes (Problem of the Month) is the only section of problems that offers a prize for a correct solution. The prize is symbolic, an annual subscription to *MATICO* itself, but the problems are quite difficult. This means, unfortunately, that not many prizes have been won.

Noticias (News) carries the results of the various regional, national and international competitions.

¿Como le hubiera ido? (How Would You Have Done?) publishes all the problems and answers for all the Colombian competitions and for the International Mathematical Olympiad (IMO) so that readers can study them in detail.

There are also three sections of reference material for students and teachers. The first, *Creciendo en matemáticas* (Growing in mathematics), is for pupils in their final years of secondary school education. Its purpose is either to explain some theme that is seldom dealt with in schools or to suggest a particularly neat way of solving problems and to describe various

1. The first Colombian magazine to propose ingenious mathematical problems and solutions was published in Bogotá in 1848 under the title 'Crónica mensual'. Unfortunately, a controversy that divided Colombian education from top to bottom led to its closure a few years later. (Universidad 'Antonio Nariño', Apartado Aereo 44564, Bogotá, D.E.)

contexts in which it has been used. 'Primeros pasos' (First steps) introduces the world of mathematics to young people who are just beginning their secondary school education. 'Curiosidades matemáticas' (Curiosities) is intended for teachers and university students. It sets problems and solutions in a historical context, recounting anecdotes about mathematicians and mathematics. The subsection entitled 'Hacia la cuarta dimensión' (Towards the fourth dimension) is the one which has had the greatest impact.

The review contains other sections that do not appear regularly, but which, from time to time, examine one of the inexhaustible number of topics that could be interesting and helpful for a young maths student. Off-prints with titles such as *El Matemático detrás* (The Mathematician Behind) and *Breviario de la matemática griega* (Breviary of Greek Mathematics) are occasionally published.

So far, the nuber of subscribers to *MATICO* has remained small, but gradually it is becoming better known and establishing itself in the educational community. One thing is clear: *MATICO* is certainly fulfilling its objective of providing the mathematics clubs with enough and sufficiently varied material for one meeting a week over two months and helping them to sustain their young members' interest.

Mathematics clubs

The newspaper's Sunday section published an article entitled 'Un Club de matemáticas en cada colegio' [A mathematics club in every secondary school]. Its purpose was to encourage the establishment of extra-curricular mathematics activities in secondary schools. The article set off such an avalanche of correspondence that the organizers of the Mathematics Olympiad were obliged to produce additional material and services for clubs. This led, in turn, to the publication of *MATICO* and the organization of workshops in various cities in the country.

In Colombia today mathematics clubs are being organized by individual secondary schools, for groups of secondary schools and in small towns. One of the most active is the mathematics club in Chiquinquirá, a city of 125,000 inhabitants. Pupils from all the city's schools attend its meetings. If often organizes workshops, inviting lecturers from universities and from cities such as Bogotá.

The mathematics club is an example of how to make the most of limited human resources and it offers institutions a marvellous opportunity to cater for students who excel in mathematics without the risk of damage to pupils without that flair.

In Colombia, education suffers from this fact that not all of the teachers in secondary schools have completed their professional training. But when a club is formed, its activities can be entrusted to the best qualified teacher or teachers. At the same time, it may be possible, as is the case, to entice university lecturers to run a school club. In the example of Chiquinquirá, the club is led by one teacher and includes students from all the schools in the town. Here, then, is a case where, with just one well-qualified person, school pupils can be provided with a scientific training that enables them to improve themselves and develop their potential to the full.

El Espectador's weekly column also provides support for people who are attempting to meet the needs of the keener pupils by starting a club for them.

Booklets

Since 1981, thirty booklets of problems, solutions and methods have been published in order to help in the task of improving and giving a new focus to mathematics education in Colombia through out-of-school activities. Most were prepared by the Olympiad Organizing Committee and by young people who represented Colombia in the IMO. Three are reprints of publications from other countries, translated by agreement into Spanish. They can be divided into five categories, each of which meets a specific need, as follows:

Problems and answers from previous competitions. Grouped by academic year, each of these publications gives an account of all the national and regional tests held in Colombia during one academic year.

The International Olympiads. Every year a booklet is published containing the questions set in the IMO and answers. The solutions are those found by the Colombian participants in the competition. A booklet containing selected problems (with answers) from the national competitions of a number of other countries is also published.

Problems for mathematics clubs. These are collections of geometry problems, algebra problems, and combinatorial and probability problems. An account of a geometry workshop, a booklet on club organization and direction and collections of problems and solutions that have appeared in the column in *El Espectador* are all available to be used as a basis for discussion at meetings of mathematics clubs, at seminars and in the course of other projects designed to improve the quality of pre-university mathematics.

Training session problems. Publications under this heading include booklets of problems involving number theory, inequalities, maxima and minima without calculus and the theory of graphs. These are the most specialized of all the publications; the problems they contain were used at training sessions for the teams that have represented Colombia in international competitions. They are designed for pupils who are intent on developing their mathematical abilities to the utmost.

Publications for the teacher. So far, there is only one booklet, *La enseñanza a través de problemas* (Teaching through problems) whose purpose is specifically to back up teachers' efforts to make problem-solving an integral part of their method of teaching. More material needs to be provided for teachers so as to enable them to improve the quality of regular courses. There are already plans for a second publication designed to meet this need.

In four short years, the production of this type of reference material has developed on a large scale and it has remedied the lack of publications in Spanish on mathematics problems. There were virtually none in 1981, even counting imported books.

Competitions

What are known today as the Colombian Mathematics Olympiads began with a local competition, organized in 1981, in the City of Bogotá by the National University of Colombia and the Universidad 'Antonio Nariño'. Soon afterwards, a body was founded to become responsible for organizing regional and national competitions in Colombia. It later assumed responsibility for the country's participation in the IMO after the first Colombian team had, with support from the Ministry of National Education, taken part in the 1981 IMO in Washington. Other important organizations, such as the *Societad Colombiana de Matemáticas* and the *Fondo Colombiano de Investigaciones Científicas y Proyectos Especiales* (Colciencias) have since become sponsors of the Colombian Mathematics Olympiads.

Purpose. It should be emphasized that, from the outset, the Colombian Mathematics Olympiads had a definite sense of purpose and a clear objective. It was decided to devise competitions that would require Colombian students and teachers to rise above the limitations of classroom teaching. Since Colombia is not a country with a solid mathematical tradition, a contest geared to the existing level of mathematical education might simply have established more firmly the country's low level of mathematical development. At the same time, one could hardly set a very difficult competition without providing the schools, teachers and pupils with the means to acquit themselves creditably in a competition of that level. This is why the Olympiads always involved more than the mere organization of a competition, as the foregoing paragraphs have made clear.

Also, the weekly correspondence inspired by the column in *El Espectador* ensures permanent contact with the eudcational community's preoccupations and interests. It provides almost instant feedback that makes it possible continuously to adjust the competitions to the public's reactions in an energetic and responsible way.

Overview of the present competitions. Perhaps the best way of describing the activities of the Colombian Olympiads is to list the events planned for the present school year. In early March, Professor Samuel Greitzer led a preparatory workshop in Bogotá for the first round of competitions, the National Classifying Test. With the assistance of several Colombian teachers, a week of lectures and problem-solving sessions was organized with two parallel programmes: one for pupils in the sixth to eighth year of schooling (first three grades of secondary education), aged 11 to 14 years; and another for pupils in the ninth to eleventh grade, aged roughly between 15 and 18. This sets the tone for the year's work, with its emphasis on preparation rather than on the competition itself.

Classifying round. The National Classifying Tests, which are set for the two levels catered for in the preparatory sessions, were held at the end of March. These are the first of a series of three rounds of tests. Each round sets a different type of question. Thus the student who moves on to the second round of the competition is faced with a more difficult test than the preceding one. Each of the three types of question used, multiple choice, short numerical answer and demonstration, assesses a different aspect of the participant's mathematical skills.

The Classifying Tests are open to all secondary school pupils. Entry formalities are organized through the schools, but individual participation by students whose schools do not take part is also permitted. For both levels, the tests comprise thirty multiple-choice questions, since this type of test is the most convenient for assessing the large numbers of participants. Each participant starts off with a total of 30 points to his or her credit. Each gains 4 points for each correct answer and loses 1 point for each incorrect answer. The possible maximum is thus 150 and the minimum 0. Because of the large number of participants, the tests are held in each school. They are supervised by the principal of the school and the teacher in charge. In some cities with regional co-ordinators, all candidates take the test in one centre. This is the case in most of the capitals of the departments.

In the first two years of the Olympiads, a single test was held for all entrants. The first level is the most important because it involves the largest number of students, teachers and schools. As the regional competition (Bogotá, 1981) showed that some students did quite well in a test of the level of the American High School Mathematics Examination (AHSME), organized by the Mathematics Association of America (MAA), it was decided to use this test in the classifying round. The first level tests for the first two years were devised in Colombia. But, from 1986, the MAA junior high school contest set has been used. The problems set for that contest are so interesting that even a student whose performance is not first-rate can enjoy the time spent working on them and be spurred on to prepare better for the next competition, especially if greater emphasis is placed on preparation than on the competition itself.

Moreover, the test helps to bring the mathematical skills in the country as a whole up to an internationally respectable level. It should, however, be pointed out that, while the results of the test in Colombia are less brilliant than those obtained in some other countries, a score of 90 or 100 points is now regarded as good. In other words, the students' compete with each other and they feel no frustration as long as they have the goal of constant improvement always before them. Two representative problems from the first round of the competition are given below:

> First level: There are six people in a photograph. If in any group of three of them there are at least two members of the same family, the number of families represented is
>
> (A) 2 (B) 3 (C) 4 (D) 5 (E) 6.
>
> Advanced level: A right-angled triangle ABC with hypotenuse AB has side AC = 15. The altitude CH divides AB into segments AH and HB, and HB = 16. The area of the triangle is
>
> (A) 120 (B) 144 (C) 150 (D) 216 (E) $144\sqrt{5}$.

Although the purpose of the classifying test is to select a number of students who then move on to the second round of the competition, it also offers its own system of prizes to encourage more students to take part.

Eliminatory round. Approximately 10 per cent of the participants in the classifying tests are invited to sit for the second round of tests. The questions asked in this round, which is held in April, are of the 'short answer' type. In other words, the problems are devised in a way that makes the answer an integer (between 0 and 1,000). The student must solve the problem and give the answer in the form of a numeral, instead of choosing from several possible sources.

The final round of the first level. At the first level, the second round is the last one. The questions are set in Colombia by members of the Problems Committee of the Olympiad Organizing Committee, and the students who do best are declared the winners of the first level Olympiads. The prizes are awarded at the same time as those which are awarded for the final round of the advanced level in May. It should be pointed out that, at this level, girls do as well as boys. In 1985, first place was won by a girl. At the same time, it is absolutely clear that the younger the competitors in Colombia are the greater the creativity they show in their approach to the problems set. In the final round of the first level, the problems are designed to encourage participants to use their ingenuity and imagination. The following is a typical problem:

> Nicholas has ten bank notes, one of each of the following values: $1,000, $500, $200, $100, $50, $20, $10, $5, $2 and $ 1. How many amounts (over 0) of less than $1,888 can Nicholas pay without needing to be given change?

The advanced level eliminatory round. The second round of the advanced level is called the eliminatory test. This is because the competitors in the final round (held in Bogotá) are selected on the basis of the second round. This test is also based on an international test devised by the MAA, which has been modified by removing some of the original problems and replacing them with others devised by the Organizing Committee of the Colombian Olympiads. Both second round tests are held in one centre in each of the cities taking part. Competitors living in isolated areas sit the test at the nearest school.

Not all the winners in this round were previously prize-winners in the classifying tests. This is because, in the second round, a different aspect of the participants' mathematical ability is being assessed.

When the whole process has been completed, those competitors possesing the broadest and most wide-ranging mathematical skills will have been selected.

The following problem is typical of those set in this round:

> An electric-light bulb is placed on top of a vertical post 6m high. Three metres away from the base of the post there is a rod than can change its

> angle in relation to the ground, keeping its base fixed at 3m from the base of the post. If the rod is 3m long, what is the maximum length of its shadow?

The final. The final of the Colombian Mathematics Olympiad is held in Bogotá. This underlines its importance as the climax of the national competition. The final takes place in two stages, each lasting 3 1/2 hours, on consecutive days. On both days the student are given a test containing eight problems whose solutions have to be 'justified', each stage in the argument being clearly set out. These questions are set by the Colombian Organizing Committee and are generally considerably easier than the problems set in the finals of national competitions in countries with more mathematical experience. It is intended that every competitor must be able to solve some of the problems set in the finals. The standard has risen a little each year and it is hoped that it will eventually be comparable to that of the national competitions in most other countries. The following two problems are typical of those set in the final round:

> Regular polygons with n and m sides respectively ($n \neq m$) can be inscribed within the same circumference so that the ratio between the areas of the polygons is n/m. Find all the pairs (n,m) that would fulfil these conditions.
>
> I is the incentre of the triangle ABC; A, I and D are collinear and D is on the side BC. Show that $AI/ID = (AB + BC)/BC$.

The prizes awarded to the winners of the Colombian Olympiads are still only symbolic: medals, diplomas, shields and other small mementoes. An attempt has been made to establish a programme of fellowships to be awarded by the Ministry of Education in recognition of the efforts made by pupils who distinguish themselves in these competitions. One of the priority objectives of out-of-school activities must be to improve the opportunities for higher education as much as possible.

Regional competitions

Regional competitions are held in October of each year (the first was held in 1982). The problems were originally set by the National Organizing Committee, but, more recently, an international test set in Australia has been used. These competitions are usually held in various towns, enabling students from small towns to show their worth in their own environment.

The competitions are set at two levels and they test ingenuity more than knowledge of mathematical theory. This slant makes them particularly attractive to the participants.

The regional contests culminate in Colombia's only team competition, called 'Mathematics Day'. Five teams are formed from the five schools in the towns which scored highest in the regional competition. In this context; 'highest' means the highest of the sum of the five highest scores of pupils from the particular school rather than the highest average of scores. The activities of the 'Mathematics Day' are divided into three categories: relay competitions (team members, one after the other, work through a set of problems; the winning team is the one that solves the greatest number of problems in the time allowed); 90-second questions (the team must solve the problem assigned to it in 90 seconds or pass it on to the next team); and lastly, a team solution of a problem in 20 minutes. 'Mathematics Day' is beginning to attract attention and interest. It encourages contacts and exchanges between schools, in much the same way as sports contests do, although on a much smaller scale.

New initiatives in out-of-school mathematics

The desire to identify and meet the needs of the educational community and to respond to its aspirations has recently led to the promotion of two new extra-curricular programmes.

Olympiads for primary schools. An alarming number of Colombian children, approximately 43 per cent of all pupils, have to repeat a year of primary schooling. The major causes are language problems (Spanish) and mathematics. The most notable consequences are dropping out of school altogether and acquiring misgivings about mathematics. Colombia's first Primary School Mathematics Olympiad in 1985 was an attempt to give mathematics education a more positive image and to shift its emphasis. Books have recently been published in connection with the competition in order to assist prospective contestants to prepare themselves for it. These publications also show teachers ways of improving their teaching methods. While primary level teachers in Colombia are unhappy about their pupils' performance, they have yet to find ways of improving it.

Ibero-American Mathematics Olympiads. In 1985, Colombia took the initiative of organizing the first Ibero-American Mathematics Olympiad in the wish to share the benefits it had felt from its own competitions, and with the desire to promote educational and scientific co-operation within this vast region which shares a common language. The Ibero-American region comprises twenty-two countries, only four of which currently participate in the IMO. As a result of the Colombian invitation, several Latin-American countries are now holding their first national mathematics Olympiads; the books published for the Colombian Olympiads (in an attempt to make good the deficiency in Spanish publications in this field) have served to support and focus competitions in other countries on our continent.

Final stock-taking

Three points have to be made in appraising out-of-school activities aimed at improving mathematics skills in Colombia.

Firstly, the increase in participation in the competitions is extremely gratifying. In 1981, all 111 participants were from the capital city, whereas, in 1985, there were 20,000 participants from all the country's departments, two intendancies and a commissioner's district (i.e. three sparsely populated territories). In 1985, the contestants sat the various tests organized as part of the primary and secondary regional and national competitions.

Secondly, out-of-school mathematics activities have generated new ventures in other sciences, such as the Colombian Physics Olympiad organized in 1985, together with related activities such as the publication of materials, science articles in the press, workshops and many others. Depending on the progress made in the intervening years of preparation and self-improvement, Colombia hopes to request an invitation to this international event in 1990.

Lastly, in recent years, Colombia has succeeded in obtaining a higher total score in the IMO than that of several countries with a longer and more solid mathematics tradition. Moreover, Colombian students have won prizes, a reward for hard work and meticulous preparation. But, most important, pupils from a country generally classified as 'third-world' have been able to compete in an international event without any embarrassment, indeed with confidence. A door towards the future has been opened.

Biographical notes

MARY FALK DE LOSADA was born in Seattle, Washington, in 1943. She was awarded a masters degree in mathematics education (MAT) from the University of Harvard in 1965 and was a Ford Foundation Fellow at the University of Illinois from 1968 to 1970. Since 1966, she has been a lecturer in the Mathematics Department of the National University of Colombia and, from 1983, Director of the Colombian Mathematics Olympiad. She is currently a member of the governing board of the World Federation of Mathematics Competitions. She has written books for secondary schools, directs the magazine *MATICO*, edits the *Boletin de Matematica* (published by the National University of Colombia) and, since November 1981, has been writing the weekly column of problems in the newspaper *El Espectador*.

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